



Heriot-Watt University
Research Gateway

On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations

Citation for published version:

Forlano, J & Trenberth, WJ 2019, 'On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations', *Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire*, vol. 36, no. 7, pp. 1987-2025. <https://doi.org/10.1016/j.anihpc.2019.07.006>

Digital Object Identifier (DOI):

[10.1016/j.anihpc.2019.07.006](https://doi.org/10.1016/j.anihpc.2019.07.006)

Link:

[Link to publication record in Heriot-Watt Research Portal](#)

Document Version:

Version created as part of publication process; publisher's layout; not normally made publicly available

Published In:

Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire

Publisher Rights Statement:

©2019 The Author(s). Published by Elsevier Masson SAS.

General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact open.access@hw.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN ●●● (●●●●) ●●●–●●●

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
ANALYSE
NON LINÉAIREwww.elsevier.com/locate/anihpc

On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations

Justin Forlano ^{a,*}, William J. Trenberth ^b^a Maxwell Institute for Mathematical Sciences, Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom^b Maxwell Institute for Mathematical Sciences, School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, United Kingdom

Received 17 December 2018; received in revised form 28 June 2019; accepted 27 July 2019

Abstract

Under certain regularity conditions, we establish quasi-invariance of Gaussian measures on periodic functions under the flow of cubic fractional nonlinear Schrödinger equations on the one-dimensional torus.

© 2019 The Author(s). Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

MSC: 35Q55

Keywords: Fractional nonlinear Schrödinger equation; Quasi-invariance; Gaussian measure

1. Introduction

1.1. Cubic fractional nonlinear Schrödinger equations

We consider the cubic fractional nonlinear Schrödinger equation (FNLS) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} i\partial_t u + (-\partial_x^2)^\alpha u = \pm |u|^2 u, \\ u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

where $u : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}$ is the unknown function. For $\alpha > 0$, let $(-\partial_x^2)^\alpha$ be the Fourier multiplier operator defined by $((-\partial_x^2)^\alpha f)^\wedge(n) := |n|^{2\alpha} \hat{f}(n)$, $n \in \mathbb{Z}$, where \hat{f} denotes the Fourier transform of f . We say (1.1) is defocusing when the sign on the nonlinearity is positive and focusing when the sign on the nonlinearity is negative.

The equation FNLS (1.1) arises in various physical settings. When $\alpha = 1$, the equation (1.1) is the cubic nonlinear Schrödinger equation (NLS) which appears as a model in the study of nonlinear optics, fluids and plasma physics; see [40] for a general survey. For $\alpha = 2$, (1.1) corresponds to the cubic fourth order NLS (4NLS) and has applications to

* Corresponding author.

E-mail addresses: j.forlano@hw.ac.uk (J. Forlano), W.J.Trenberth@sms.ed.ac.uk (W.J. Trenberth).

<https://doi.org/10.1016/j.anihpc.2019.07.006>

0294-1449/© 2019 The Author(s). Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license

(<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

the study of solitons in magnetic materials [22,42]. FNLS with non-local dispersion $\frac{1}{2} < \alpha < 1$ arises in continuum limits of long-range lattice interactions [23]. When $\alpha \leq \frac{1}{2}$, the equation FNLS (1.1) is no longer dispersive. However, the cubic nonlinear half-wave equation, corresponding to $\alpha = \frac{1}{2}$, emerges in the study of wave turbulence [26,6], gravitational collapse [12,14] and has been well-studied analytically [17,25,16]. We also refer to [41] for a study of (1.1) when $\frac{1}{3} < \alpha < \frac{1}{2}$. In the following, we focus on studying FNLS (1.1) for $\alpha > \frac{1}{2}$ where dispersion is present.

The well-posedness theory of (1.1) in the L^2 -based Sobolev spaces $H^\sigma(\mathbb{T})$ crucially depends upon the strength of the dispersion; namely, if $\alpha \geq 1$ or $\frac{1}{2} < \alpha < 1$. In [1], Bourgain proved local well-posedness of NLS in $L^2(\mathbb{T})$, which immediately extends to global well-posedness in $L^2(\mathbb{T})$ as a consequence of mass conservation:

$$M(u)(t) = \int_{\mathbb{T}} |u(t, x)|^2 dx = M(u)(0) \quad \text{for all } t \in \mathbb{R}. \quad (1.2)$$

A persistence-of-regularity argument then implies global well-posedness of NLS in $H^\sigma(\mathbb{T})$ for any $\sigma \geq 0$. This result is sharp in the sense that NLS is ill-posed if $\sigma < 0$. More precisely, the solution map¹ $\Phi : u_0 \in H^\sigma(\mathbb{T}) \mapsto u \in C([-T, T]; H^\sigma(\mathbb{T}))$, if it even exists in view of the non-existence of solutions in [20], is discontinuous [27] (see also [5,37,9,30,24]). In [33, Appendix A], the 4NLS was shown to be globally well-posed in $H^\sigma(\mathbb{T})$ for any $\sigma \geq 0$. In Appendix B of this paper, we extend this global well-posedness result to FNLS (1.1) for any $\alpha > 1$.

The well-posedness situation for FNLS (1.1) is somewhat less complete in the setting $\frac{1}{2} < \alpha < 1$. In [8], Cho, Hwang, Kwon and Lee proved local well-posedness of FNLS (1.1) in $H^\sigma(\mathbb{T})$ for $\sigma \geq \frac{1-\alpha}{2}$ by a contraction mapping argument; see also [11]. Following the argument in [33, Appendix A], we can show the solution map for FNLS (1.1) fails to be locally uniformly continuous in $H^\sigma(\mathbb{T})$ for any $\sigma < 0$. Thus we can not construct solutions below $L^2(\mathbb{T})$ using a contraction mapping argument. See also [9].

As for global well-posedness, the flow of FNLS (1.1) conserves the energy $H(u)$; that is,

$$H(u)(t) = \frac{1}{2} \int_{\mathbb{T}} \left| (-\partial_x^2)^{\frac{\alpha}{2}} u(t, x) \right|^2 dx \pm \frac{1}{4} \int_{\mathbb{T}} |u(t, x)|^4 dx = H(u)(0). \quad (1.3)$$

In the defocusing case, the energy controls the H^α -norm and hence energy conservation can be used to globalise (in time) all solutions with regularity at or above the energy space, that is, for when $\sigma \geq \alpha$. This result also holds in the focusing case, as the H^α -norm can be controlled in terms of both the energy and the mass by using the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \lesssim \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{L^2}^{\frac{1}{\alpha}} \|u\|_{L^2}^{4-\frac{1}{\alpha}} \quad (1.4)$$

for $\alpha \geq \frac{1}{4}$. Below the energy space $H^\alpha(\mathbb{T})$, global well-posedness of FNLS (1.1) (for both defocusing and focusing nonlinearities) in $H^\sigma(\mathbb{T})$ was obtained for $\sigma > \frac{10\alpha+1}{12}$ [11] by using the high-low frequency decomposition of Bourgain [4].

In summary, the flow of FNLS (1.1) is well-defined under the following conditions:

Proposition 1.1 (Well-posedness of the cubic FNLS (1.1) in $H^\sigma(\mathbb{T})$ [1,8,11,33]).

- (i) Let $\alpha \geq 1$. Then, the cubic FNLS (1.1) is globally well-posed in $H^\sigma(\mathbb{T})$ for $\sigma \geq 0$.
- (ii) Let $\frac{1}{2} < \alpha < 1$. Then, the cubic FNLS (1.1) is locally well-posed in $H^\sigma(\mathbb{T})$ for $\sigma \geq \frac{1-\alpha}{2}$. Moreover, the cubic FNLS (1.1) is globally well-posed for $\sigma > \frac{10\alpha+1}{12}$.

In the following, we make no distinction between the defocusing or focusing nature of (1.1) and henceforth we assume that (1.1) is defocusing. For future use, we define $\Phi(t) : u_0 \in H^\sigma(\mathbb{T}) \mapsto u(t) \in H^\sigma(\mathbb{T})$ to be the solution map of FNLS (1.1) (when it exists) at time t .

¹ For clarity of presentation, we neglect to explicitly show the dependence of the solution map Φ on α . Unless otherwise stated, the precise value of α will be clear from the context.

1.2. Main result

Our goal in this paper is to study the transport property of Gaussian measures on periodic functions under the flow of FNLS (1.1). Given $s \in \mathbb{R}$, we define the Gaussian measure μ_s to be the induced probability measure under the map²:

$$\omega \in \Omega \longmapsto u^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}, \quad (1.5)$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n \in \mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables, i.e. $\text{Var}(g_n) = 2$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Formally, μ_s has density

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}_n|^2} d\hat{u}_n.$$

A computation shows that the random distribution (1.5) belongs to $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ and not to $H^{s-\frac{1}{2}}(\mathbb{T})$ almost surely. It follows that the Gaussian measure μ_s is supported on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{s-\frac{1}{2}}(\mathbb{T})$ for any $\varepsilon > 0$. Thus, in order to discuss the transport property of these measures under the flow of (1.1), Proposition 1.1 restricts us to the range:

$$s > \max\left(\frac{1}{2}, 1 - \frac{\alpha}{2}\right), \quad (1.6)$$

which ensures there exist well-defined dynamics within the support of μ_s . We now state the definition of quasi-invariant measures: given a measure space (X, μ) , we say that μ is quasi-invariant under a transformation $T : X \rightarrow X$ if the push-forward measure $T_*\mu = \mu \circ T^{-1}$ and μ are mutually absolutely continuous with respect to each other.

The quasi-invariance of Gaussian measures supported on periodic functions under the flow of 4NLS was recently studied by Oh and Tzvetkov [33] and Oh, Sosoe and Tzvetkov [31]. See also Remark 1.5 and the recent work [32] on Schrödinger-type equations. Our main goal is to extend these quasi-invariance results to more general values of dispersion α . Thus, in this direction we establish the following:

Theorem 1.2. *Let $s \in \mathbb{R}$ and $\alpha > \frac{1}{2}$ be such that*

- (i) $s > \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right)$ if $\alpha \geq 1$, or
- (ii) $s > \frac{10\alpha+7}{12}$ if $\frac{1}{2} < \alpha < 1$.

Then, the Gaussian measure μ_s is quasi-invariant under the flow of the cubic FNLS (1.1). More precisely, given any measurable set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ satisfying $\mu_s(A) = 0$, we have $\mu_s(\Phi(-t)(A)) = 0$ for every $t \in \mathbb{R}$.

Our proof of quasi-invariance of μ_s in the case $\frac{1}{2} < \alpha < 1$ also holds for any

$$\min\left(1, \frac{11}{6} - \alpha\right) < s \leq \frac{10\alpha+7}{12}. \quad (1.7)$$

The restriction in Theorem 1.2 (ii) is due to a lack of globally well-defined dynamics within the support of μ_s for s satisfying (1.7) (see Proposition 1.1 (ii)). However, our arguments in this paper allow us to recover the following *local-in-time quasi-invariance* result:

Theorem 1.3 (Local-in-time quasi-invariance). *Let $\frac{1}{2} < \alpha < 1$ and s satisfy (1.7). Then, for every $R > 0$, there exists $T > 0$ such that for every measurable*

$$A \subset \{u \in H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) : \|u\|_{H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})} < R\}$$

satisfying $\mu_s(A) = 0$, we have $\mu_s(\Phi(-t)(A)) = 0$ for every $t \in [-T, T]$.

² From now on, we drop the factor 2π as it plays no role in our analysis.

We conclude this subsection with a few remarks.

Remark 1.4. We note that any lowering of the global well-posedness regularity threshold, to say σ_0 , for the FNLS (1.1) as stated in Proposition 1.1 (ii), will immediately imply a corresponding improvement to Theorem 1.2 (ii) and Theorem 1.3. That is, we can ‘upgrade’ from local-in-time quasi-invariance to quasi-invariance as in Theorem 1.2, provided

$$\sigma_0 - \frac{1}{2} > \min\left(1, \frac{11}{6} - \alpha\right).$$

This should be contrasted with the local-in-time quasi-invariance result in [38, Theorem 1.5] for the focusing quintic NLS on \mathbb{T} . In that setting, a global flow does not exist in view of the presence of finite-time blow-up solutions (see for example [29]). Thus, it is impossible to remove the ‘local-in-time’ restriction.

Remark 1.5. In the works of Bourgain [3] and Zhidkov [44], it was proven that for each $k \in \mathbb{N}$, the NLS has an invariant weighted Gaussian measure ρ_k which is mutually absolutely continuous with the Gaussian measure μ_k . Thus, the invariance of the measures ρ_k imply the quasi-invariance of the Gaussian measures μ_k for each $k \in \mathbb{N}$. See also [34] for further discussion. Theorem 1.2 extends these results to quasi-invariance of Gaussian measures μ_s under the flow of NLS to (not necessarily integer) regularities $s > \frac{5}{6}$.

1.3. Methodology and discussion

Theorem 1.2 is an addition to a recent program initiated by Tzvetkov [43] based on understanding the role dispersion has on the transport properties of Gaussian measures under the flow of nonlinear Hamiltonian PDEs; see also [33, 35, 31, 32, 38, 19]. Within the context of abstract Wiener space, the classical work of Ramer [39] (see also [7]) studied the quasi-invariance of Gaussian measures under general nonlinear transformations. In the setting of d -dimensional nonlinear Hamiltonian PDEs, Ramer’s result can be interpreted as requiring a $(d + \varepsilon)$ -degree of smoothing on the nonlinear part of the flow generated by a given PDE [43]. Recently, Tzvetkov [43] introduced a general methodology for proving quasi-invariance of Gaussian measures under the flow of nonlinear Hamiltonian PDE which does not appeal to Ramer’s result. In that paper, Tzvetkov studied the generalised BBM equation and crucially made use of the explicit smoothing present on the nonlinearity to prove quasi-invariance of Gaussian measures μ_s . However, in our case of FNLS (1.1), there is no explicit smoothing on the nonlinearity. To overcome this and reveal the necessary smoothing, we employ gauge transformations and normal form reductions on (1.1) (see Section 3).

Our proof of Theorem 1.2 and Theorem 1.3 is split into two parts. In the first, we employ the recent argument in [38] (see Method 3 below) to obtain quasi-invariance for all $\alpha > \frac{1}{2}$, for some range of regularities s . We then apply the argument in [19] (see Method 4 below) to improve upon the previous regularity restriction for $\alpha > \frac{5}{6}$. We now go over the recent developments in the study of quasi-invariance of Gaussian measures under the flow of nonlinear Hamiltonian PDEs. Note that, in order to rigorously justify the methods below, it is necessary to consider a suitably truncated version of (1.1) (see for example (3.2) and (3.6)). Furthermore, for the sake of discussion, we restrict to the one-dimensional case.

• **Method 1: (‘Ramer’s argument’)** The first method is to directly verify the hypothesis of Ramer’s result [39] on quasi-invariance of Gaussian measures under general nonlinear transformations. In the one-dimensional context, this essentially reduces to demonstrating a $(1 + \varepsilon)$ -degree of smoothing for the nonlinear part of the flow. This approach was applied in [43, 33, 32].

‘Energy methods:’

• **Method 2:** Introduced by Tzvetkov [43], the second method involves both nonlinear PDE techniques and stochastic analysis. We give an overview of the method here; see also [34]. Let Ψ be the flow of a given PDE. Given a measurable set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ satisfying $\mu_s(A) = 0$, we aim to show

$$\mu_s(\Psi(t)(A)) = Z_s^{-1} \int_{\Phi(t)(A)} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = 0 \text{ for all } t \in \mathbb{R}, \quad (1.8)$$

by obtaining a differential inequality of the form

$$\frac{d}{dt} \mu_s(\Psi(t)(A)) \leq Cp^\beta \{\mu_s(\Psi(t)(A))\}^{1-\frac{1}{p}}, \quad (1.9)$$

where $0 \leq \beta \leq 1$ and $p < \infty$. Then, applying Yudovich's argument (or a variant of, see [43,35]) to (1.9) implies (1.8) for small times. The argument can then be iterated to give (1.8) for all times. Thus, matters reduce to obtaining (1.9). By Liouville's theorem and the bijectivity of the flow Ψ , we have the following 'change of variables' formula (Lemma 5.6):

$$\mu_s(\Psi(t)(A)) = Z_s^{-1} \int_A e^{-\frac{1}{2} \|\Psi(t)u\|_{H^s}^2} du \text{ for all } t \in \mathbb{R}.$$

Taking a time derivative, evaluating at a fixed $t_0 \in \mathbb{R}$ and using the group property of the flow, $\Psi(t+t_0) = \Psi(t)\Psi(t_0)$, we obtain

$$\begin{aligned} \left. \frac{d}{dt} \mu_s(\Psi(t)(A)) \right|_{t=t_0} &= -\frac{1}{2} Z_s^{-1} \int_{\Psi(t_0)(A)} \frac{d}{dt} \left(\|\Psi(t)(u)\|_{H^s}^2 \right) e^{-\frac{1}{2} \|\Psi(t)u\|_{H^s}^2} \Big|_{t=0} du \\ &\leq C \left\| \frac{d}{dt} \left(\|\Phi(t)(u)\|_{H^s}^2 \right) \right\|_{t=0} \Big\|_{L^p(\mu_s)} \{\mu_s(\Psi(t_0)(A))\}^{1-\frac{1}{p}}. \end{aligned} \quad (1.10)$$

Thus, we lead to the following energy estimate (with smoothing):

$$\left. \frac{d}{dt} \left(\|\Psi(t)(u)\|_{H^s}^2 \right) \right|_{t=0} \leq C(\|u\|_B) \|u\|_X^\theta, \quad (1.11)$$

where $\theta \leq 2$. Here, we have the freedom to choose the X -norm above provided it captures the regularity of the random distribution (1.5) almost surely; for example, we may take $X = H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, the Bessel potential space $W^{s-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T})$ or the Fourier-Lebesgue space $\mathcal{FL}^{s-\varepsilon, \infty}(\mathbb{T})$ (see (1.18)). On the other hand, we must choose the weaker B -norm so that it can be controlled in terms of conserved quantities of the given PDE. The inequality (1.9) then follows from (1.10), (1.11) and estimates on higher moments of Gaussian random variables (see (5.17)). Indeed, the reduction to time $t=0$ in the above analysis allows us to use stochastic tools on the explicit random distribution (1.5).

For the generalised BBM equation, Tzvetkov [43] was able to obtain a suitable energy estimate of the form (1.11). Unfortunately, for general dispersive PDE such an estimate does not always hold. The key modification is to instead consider a 'modified energy' E of the form

$$E(u) = \|u\|_{H^s}^2 + \text{correction terms}$$

and obtain the following estimate (with smoothing):

$$\left| \frac{d}{dt} E(\Psi(t)(u)) \right|_{t=0} \leq C(\|u\|_B) \|u\|_X^\theta. \quad (1.12)$$

Now, provided we show the measure ρ_s with density

$$d\rho_s = Z_s^{-1} e^{-E(u)} du \quad (1.13)$$

can be normalised into a probability measure, we can repeat the above argument for ρ_s and conclude the quasi-invariance of ρ_s under the flow Ψ . Finally, we appeal to the mutual absolute continuity of ρ_s and μ_s to conclude the quasi-invariance for μ_s under the flow Ψ . To summarise, Method 2 requires two crucial ingredients: (i) a modified energy estimate of the form (1.12) and (ii) the construction of the weighted Gaussian measure ρ_s in (1.13).

• **Method 3:** Introduced by Planchon, Tzvetkov and Visciglia [38], where they studied the quasi-invariance of Gaussian measures under the flow of the (super-)quintic NLS on \mathbb{T} , the third approach is similar in spirit to Method 2. The fundamental feature of this method is the use of deterministic growth bounds on the $H^{s-\frac{1}{2}-\varepsilon}$ -norm of solutions (see Proposition 4.6), so that the analysis can be restricted to a closed ball $B_R \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$. The benefit of this idea over Method 2 is that we require a softer energy estimate:

$$\left| \frac{d}{dt} E(\Psi(t)(u)) \right| \leq C \left(1 + \|\Psi(t)(u)\|_{H^{s-\frac{1}{2}-\varepsilon}}^k \right) \quad (1.14)$$

for some $k \geq 0$. We then use (1.14) and the growth bound on solutions to show, for any $A \subset B_R$ with $\mu_s(A) = 0$, we have

$$\frac{d}{dt} \tilde{\rho}_s(\Psi(t)(A)) \leq C(R, T)^k \tilde{\rho}_s(\Psi(t)(A)) \quad \text{for all } t \in [0, T],$$

where $\tilde{\rho}_s$ is the measure with density

$$d\tilde{\rho}_s = e^{-E(u)} du.$$

Gronwall's inequality and soft arguments then imply $\tilde{\rho}_s(\Psi(t)(A)) = 0$ and hence $\mu_s(\Psi(t)(A)) = 0$ for every $A \subset B_R$. We then take $R \rightarrow \infty$ to obtain the quasi-invariance of μ_s under the flow Ψ . Two further differences to Method 2 are: (i) there is no need to reduce to time $t = 0$ to access stochastic tools and (ii) we do not need to normalise the measure $\tilde{\rho}_s$. Notice that the above argument works even when the flow is only locally-in-time well-defined, which leads to a local-in-time quasi-invariance result as in Theorem 1.3 (see also [2, p. 28]).

• **Method 4:** This approach combines aspects of Methods 2 and 3 and was introduced by Gunaratnam, Oh, Tzvetkov and Weber [19] for handling the cubic nonlinear wave equation (NLW) on \mathbb{T}^3 . Namely, by arguing locally within $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ and returning the analysis to time $t = 0$, one needs an even softer energy estimate taking the following form:

$$\left| \frac{d}{dt} E(\Psi(t)(u)) \right|_{t=0} \leq C(\|u\|_{H^{s-\frac{1}{2}-\varepsilon}}) \|u\|_X^\theta, \quad (1.15)$$

for $\theta \leq 2$ and where the X -norm may be chosen as in Method 2. Analogously to Method 2, we must also construct a suitable auxiliary probability measure adapted to the modified energy.

In the following, we survey how each of the above methods may be implemented within the context of FNLS (1.1).

In practice, Method 1 is often less applicable than Methods 2, 3 and 4 as it requires a $(1 + \varepsilon)$ -amount of smoothing on the nonlinear part of the flow. In our situation of FNLS (1.1) with $\alpha \geq 1$, we could demonstrate $(2\alpha - 1)$ -degrees of nonlinear smoothing, provided $s > 1$. Hence, Method 1 yields quasi-invariance of Gaussian measures μ_s under the flow of (1.1), provided that $\alpha > 1$ and $s > 1$. For FNLS (1.1) with $\frac{1}{2} < \alpha \leq 1$, a $(1 + \varepsilon)$ -degree of nonlinear smoothing is not expected [13] and thus we do not know at this point if Ramer's argument can be applied in this case. In addition, Methods 2, 3 and 4 usually give lower regularity restrictions compared to using Ramer's argument (e.g. [31]). We found this to indeed be the case for FNLS (1.1). For this reason, we do not present Method 1 here.

For application to FNLS (1.1), it turns out that Method 3 gives an improved result in terms of regularity over Method 2. Indeed, in terms of the energy estimate itself, the rigidity in the choice of the B -norm in (1.12) leads to far less flexibility compared to the energy estimate (1.14) in Method 3. In the regime $\frac{1}{2} < \alpha < 1$, we established an energy estimate of the form (1.12) with $B = H^\alpha(\mathbb{T})$ and $X = H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$. Thus, energy conservation (1.3) immediately places the regularity restriction $s > \alpha + \frac{1}{2}$, in this use of Method 2. This restriction is unnatural since it goes against our intuition that greater dispersion gives a lower regularity threshold. In this paper, we use Method 3 which allows us to remove the restrictions coming from using conservation laws and thus lower the regularity threshold.

We now describe our application of Method 3. The main goals are to establish (i) a suitable modified energy (see (4.5)) and (ii) a corresponding energy estimate of the form (1.14) (see Proposition 4.1). For this purpose, we apply gauge transformations to convert (1.1) into a form more amenable to apply the normal form reductions used to define the modified energy (see Sections 3 and 4). In this approach, the phase function

$$\phi(\bar{n}) = |n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n|^{2\alpha} \quad (1.16)$$

naturally arises as the source of dispersion. In order to exploit this for a smoothing benefit, which is required to achieve (ii), we crucially rely on the following lower bound: for $\alpha > \frac{1}{2}$, we have

$$|\phi(\bar{n})| \gtrsim |n - n_1| |n - n_3| n_{\max}^{2\alpha-2} \quad \text{when} \quad n = n_1 - n_2 + n_3.$$

Here, $n_{\max} := \max(|n_1|, |n_2|, |n_3|, |n|)$. This lower bound first appeared in the setting $\frac{1}{2} < \alpha \leq 1$ in [11]. With minor modifications, its proof extends easily to the case $\alpha > 1$; see Lemma 2.3 and Appendix A. It can be viewed as a replacement of the explicit factorisations available for the phase function (1.16) of NLS

$$\phi(\bar{n}) = n_1^2 - n_2^2 + n_3^2 - n^2 = -2(n - n_1)(n - n_3) \quad \text{when} \quad n = n_1 - n_2 + n_3,$$

and 4NLS (see [33, Lemma 3.1]). Following the argument in [38], we obtain quasi-invariance of Gaussian measures μ_s under the flow of (1.1) for

$$(i) s > 1, \text{ when } \frac{1}{2} < \alpha \leq 1, \quad \text{and} \quad (ii) s > s_\alpha, \text{ for some } s_\alpha \leq 1, \text{ when } \alpha > 1. \quad (1.17)$$

See (4.7) for a precise statement of s_α .

Our next goal is to attempt to lower the regularity restriction from Method 3 by using Method 4. This requires us to construct a suitable weighted Gaussian measure (Subsection 5.2) and establish an effective energy estimate of the form (1.15). In establishing the energy estimate, we have some freedom in the choice of the X -norm. One choice is the Hölder-Besov norm as used in [19]. Since we work intimately on the Fourier side, we use the *Fourier-Lebesgue* $X = \mathcal{FL}^{\sigma, \infty}(\mathbb{T})$ -norm for $\sigma < s$. Here, given $q \geq 1$ and $s \in \mathbb{R}$, the Fourier-Lebesgue $\mathcal{FL}^{s, q}(\mathbb{T})$ -norm is defined by:

$$\|f\|_{\mathcal{FL}^{s, q}(\mathbb{T})} := \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^q(\mathbb{Z})}. \quad (1.18)$$

It is easy to check that the random distribution in (1.5) belongs almost surely to $\mathcal{FL}^{\sigma, \infty}(\mathbb{T})$ for any $\sigma < s$ (Lemma 5.5). Moreover, Hölder's inequality implies the embedding

$$H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T}) \supset \mathcal{FL}^{\sigma, \infty}(\mathbb{T}) \quad (1.19)$$

for σ sufficiently close to s . This fact allows us to further relax the energy estimate we obtained in Method 3 (see Proposition 5.1). We then follow the argument in [19] to conclude quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1) for the following regularities:

$$(i) \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right) < s \leq 1, \text{ when } \alpha \geq 1 \text{ and} \quad (1.20)$$

$$(ii) \frac{11}{6} - \alpha < s \leq 1, \text{ when } \frac{5}{6} < \alpha < 1.$$

Notice that in (1.20), we improve the regularity restriction (1.17) we obtained using Method 3 only when $\alpha > \frac{5}{6}$. The reason for this is that our use of the stronger $\mathcal{FL}^{\sigma, \infty}$ -norm in the energy estimate for Method 4 (Proposition 5.1) yields a regularity gain over the energy estimate in Method 3 (Proposition 4.1) provided $\alpha > \frac{5}{6}$. Furthermore, the upper bound $s \leq 1$ in (1.20) is necessary for our construction of the weighted Gaussian measure.

Remark 1.6. When $\alpha \geq \frac{7}{6}$, the regularity restriction in Theorem 1.2 (i) achieves the largest range of $s > \frac{2}{3}$. In particular, when $\alpha = 2$, this improves upon the result in [33] of $s > \frac{3}{4}$. However, as remarked in [32], this same result of $s > \frac{2}{3}$ for 4NLS could be obtained by using Method 1 and an additional novel gauge transformation introduced in that same paper. For FNLS (1.1) with $\alpha > 1$ (and large enough), we expect the optimal result $s > \frac{1}{2}$ could be obtained by using a finer modified energy arising from an infinite sequence of normal form reductions. See [31] where this approach led to the optimal result for 4NLS.

Recently, Oh, Tzvetkov and Wang [36] established the invariance of the (Gibbs-type) measure μ_0 under the flow of the (renormalised) 4NLS; namely, (3.1) for $\alpha = 2$. Their analysis seems to also extend to the (renormalised) FNLS (3.1) for some $1 < \alpha < 2$. Thus, in the presence of naturally associated invariant measures, quasi-invariance may persist for some regularities below what (1.6) suggests.

Remark 1.7. In Theorem 1.2 and Theorem 1.3, we studied the quasi-invariance of Gaussian measures under the flow of FNLS (1.1) for $\alpha > \frac{1}{2}$. A natural question would be to study the transport property of Gaussian measures under the flow of FNLS (1.1) with $\alpha = \frac{1}{2}$. We recall this corresponds to the non-dispersive half-wave equation

$$i\partial_t u + |\partial_x u|u = |u|^2 u. \quad (1.21)$$

In this case, we expect the Gaussian measures to not be quasi-invariant under the flow of (1.21). To give some credence to this, we observe that it was shown in [31] that Gaussian measures μ_s are not quasi-invariant under the flow of the *dispersionless* equation

$$i\partial_t u = |u|^2 u. \quad (1.22)$$

Given a solution u to (1.22), the change of variables

$$u(t, x) \longmapsto u(t, x - t)$$

implies μ_s is not quasi-invariant under the flow of

$$i\partial_t u + i\partial_x u = |u|^2 u$$

which closely resembles (1.21). The proof of the non quasi-invariance under (1.22) in [31] heavily makes use of the explicit solution formula

$$u(t, x) = e^{-it|u(0,x)|^2} u(0, x).$$

Unfortunately, there is no such solution formula for the half-wave equation (1.21). Hence, at this point, we do not know how to conclude non quasi-invariance of Gaussian measures under the flow of (1.21). It would also be of interest to consider the transport properties of Gaussian measures under the flow of other dispersionless PDE, such as the cubic Szëgo equation:

$$i\partial_t u = \mathbf{P}_{\geq 0}(|u|^2 u),$$

where $\mathbf{P}_{\geq 0}$ is the projection onto non-negative frequencies $\{n : n \geq 0\}$. See [15] and the references therein for more on the cubic Szëgo equation.

Remark 1.8. It would be of interest to study how our approach in this paper may extend to higher order nonlinearities, say, for the quintic nonlinearity $|u|^4 u$. For instance, the relevant phase function is now

$$|n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n_4|^{2\alpha} + |n_5|^{2\alpha} - |n|^{2\alpha}, \quad (1.23)$$

which is restricted to the hyperplane $n_1 - n_2 + n_3 - n_4 + n_5 = n$. When $\alpha = 1$, there is no factorisation for (1.23). Therefore, an appropriate analogue of Lemma 2.3 is not clear (and likewise for Lemma 2.2). However, even if such results were proved, the formulation of the modified energies and the appropriate nonlinear estimates would still have to be verified. We note that the method in [38] introduces a modified energy functional which is not derivable from differentiation by parts, and hence their analysis is not based on factorisations of the phase function (1.23).

2. Preliminary estimates

In this section we record some elementary estimates that will be useful in the coming analysis. The first result we need is the double mean value theorem (DMVT) from [10, Lemma 2.3].

Lemma 2.1 (DMVT). *Let $\xi, \eta, \lambda \in \mathbb{R}$ and $f \in C^2(\mathbb{R})$. Then, we have*

$$f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi) = \lambda \eta \int_0^1 \int_0^1 f''(\xi + s\lambda + t\eta) ds dt.$$

We have the following consequence of DMVT:

Lemma 2.2. *Fix $s > 1$ and let $n_1, n_2, n_3, n \in \mathbb{Z}$ be such that $n = n_1 - n_2 + n_3$. Then, we have*

$$\left| \langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n \rangle^{2s} \right| \lesssim |n - n_1| |n - n_3| \langle n_{\max} \rangle^{2s-2},$$

where $n_{\max} = \max(|n_1|, |n_2|, |n_3|, |n|)$ and the implicit constant depends only on s .

Proof. This is a simple application of DMVT upon setting $n_1 = \xi + \eta + \lambda$, $n_2 = \xi + \eta$, $n = \xi + \lambda$ and $n_3 = \xi$. \square

The next lemma states a crucial lower bound on the phase function $\phi(\bar{n})$ of (1.16) which we use repeatedly throughout. It was proved for the case $\frac{1}{2} < \alpha \leq 1$ in [11]. Their proof easily extends to the case $\alpha > 1$; see Appendix A.

Lemma 2.3. Fix $\alpha > \frac{1}{2}$ and let $n_1, n_2, n_3, n \in \mathbb{Z}$ be such that $n = n_1 - n_2 + n_3$. Then, we have

$$\begin{aligned} |\phi(\bar{n})| &\gtrsim |n - n_1| |n - n_3| (|n - n_1| + |n - n_3| + |n|)^{2\alpha-2} \\ &\gtrsim |n - n_1| |n - n_3| n_{\max}^{2\alpha-2} \end{aligned}$$

where $n_{\max} = \max(|n_1|, |n_2|, |n_3|, |n|)$ and the implicit constant depends only on α .

We next state a useful summing estimate, a proof of which can be found in, for example, [18, Lemma 4.2].

Lemma 2.4. If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then we have

$$\begin{aligned} \sum_n \frac{1}{\langle n - k_1 \rangle^\beta \langle n - k_2 \rangle^\gamma} &\lesssim \frac{\varphi_\beta(k_1 - k_2)}{\langle k_1 - k_2 \rangle^\gamma}, \\ \int_{\mathbb{R}} \frac{1}{\langle x - k_1 \rangle^\beta \langle x - k_2 \rangle^\gamma} dx &\lesssim \frac{\varphi_\beta(k_1 - k_2)}{\langle k_1 - k_2 \rangle^\gamma}, \end{aligned}$$

where

$$\varphi_\beta(k) := \sum_{1 \leq |n| \leq |k|} \frac{1}{|n|^\beta} \sim \begin{cases} 1, & \text{if } \beta > 1, \\ \log(1 + \langle k \rangle), & \text{if } \beta = 1, \\ \langle k \rangle^{1-\beta}, & \text{if } \beta < 1. \end{cases}$$

Finally, we will require the following fact from elementary number theory [21]: Given $n \in \mathbb{N}$ and any $\delta > 0$, there exists a constant $C_\delta > 0$ such that the number of divisors $d(n)$ of n satisfies

$$d(n) \leq C_\delta n^\delta. \quad (2.1)$$

3. Reformulation of FNLS

In this section, we reformulate FNLS (1.1) into a more amenable form for the normal form reductions in the next section. Given $t \in \mathbb{R}$, we consider the gauge transform \mathcal{G}_t on $L^2(\mathbb{T})$ defined by

$$\mathcal{G}_t[f] = e^{2it \int |f|^2 dx} f,$$

where $\int_{\mathbb{T}} f(x) dx := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx$. Furthermore, given $u \in C(\mathbb{R}; L^2(\mathbb{T}))$, we define \mathcal{G} by

$$\mathcal{G}[u](t) := \mathcal{G}_t[u(t)].$$

It is easy to check that \mathcal{G} is invertible with inverse

$$\mathcal{G}^{-1}[u](t) = \mathcal{G}_{-t}[u(t)].$$

Now, let $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ be a solution to (1.1) and define v by

$$v(t) = \mathcal{G}[u](t).$$

Then it follows from mass conservation (1.2) that v satisfies

$$i \partial_t v + (-\partial_x^2)^\alpha v = \left(|v|^2 - 2 \int_{\mathbb{T}} |v|^2 dx \right) v. \quad (3.1)$$

Namely, v satisfies (1.1) but with a more favourable nonlinearity. We define $\Phi(t) : u_0 \in H^\sigma(\mathbb{T}) \mapsto v(t) \in H^\sigma(\mathbb{T})$ to be the solution map of (3.1) (when it exists) at time t .

In order to make the following calculations secure, we consider the following truncated equation:

$$i \partial_t v + (-\partial_x^2)^\alpha v = \mathbf{P}_{\leq N} \left[\left(|\mathbf{P}_{\leq N} v|^2 - 2 \int_{\mathbb{T}} |\mathbf{P}_{\leq N} v|^2 dx \right) \mathbf{P}_{\leq N} v \right]. \quad (3.2)$$

Here, $\mathbf{P}_{\leq N}$ is the projection onto frequencies $\{n : |n| \leq N\}$ for $N \in \mathbb{N}$. We let $\Phi_N(t)$ denote the solution map of (3.2) at time t (when it exists).

To exploit the dispersive nature of (3.1), we will need another gauge transform. We define the *interaction representation* of v as

$$w(t) = S(-t)v(t), \quad (3.3)$$

where $S(t) = e^{it(-\partial_x^2)^\alpha}$. On the Fourier side, we have³

$$\widehat{w}_n(t) = e^{-it|n|^{2\alpha}} \widehat{v}_n(t).$$

Then, the equation (3.1) becomes the following equation for the Fourier coefficients $\{\widehat{w}_n\}_{n \in \mathbb{Z}}$:

$$\partial_t \widehat{w}_n = -i \sum_{\Gamma(n)} e^{it\phi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} + i|\widehat{w}_n|^2 \widehat{w}_n, \quad (3.4)$$

where the phase function $\phi(\bar{n})$ and the plane $\Gamma(n)$ are given by

$$\phi(\bar{n}) = \phi(n_1, n_2, n_3, n) = |n_1|^{2\alpha} - |n_2|^{2\alpha} + |n_3|^{2\alpha} - |n|^{2\alpha}$$

and

$$\Gamma(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\}. \quad (3.5)$$

Similarly, the truncated equation (3.2) becomes the following equation for the Fourier coefficients $\{\widehat{w}_n\}_{n \in \mathbb{Z}}$:

$$\partial_t \widehat{w}_n = \mathbf{1}_{|n| \leq N} \left[-i \sum_{\Gamma_N(n)} e^{it\phi(\bar{n})} \widehat{w}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{w}_{n_3} + i|\widehat{w}_n|^2 \widehat{w}_n \right], \quad (3.6)$$

where the plane $\Gamma_N(n)$ is given by

$$\Gamma_N(n) = \Gamma(n) \cap \{(n_1, n_2, n_3) : |n_j| \leq N, j = 1, 2, 3\}.$$

From now on, for ease of notation, we will typically ignore the ‘hats’ on the Fourier coefficients.

The following lemma shows that it suffices to prove the quasi-invariance of μ_s under the flow of (3.1).

Lemma 3.1. *The following is true:*

- (i) Let $s > \frac{1}{2}$. Then, for any $t \in \mathbb{R}$, the Gaussian measure μ_s is invariant under the map \mathcal{G}_t .
- (ii) Let (X, μ) be a measure space and suppose that T_1 and T_2 are maps from X to itself such that μ is quasi-invariant under T_1 and is quasi-invariant under T_2 . Then, μ is quasi-invariant under the composition $T_1 \circ T_2$.

For a proof of these, see [33, Lemmas 4.4 and 4.5]. We note in the particular case $s = 1$, (i) follows from the results in [28].

4. Quasi-invariance for $\alpha > \frac{1}{2}$

In this section, we present part of the proof of Theorem 1.2 and Theorem 1.3 by applying the argument in [38] (Method 3). Namely, we establish quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1) for regularities s given in (1.17). We begin in Subsection 4.1 by deriving a suitable modified energy and obtaining the key energy estimate of the form (1.14). Then, in Subsection 4.2, we use this energy estimate to conclude the quasi-invariance of μ_s .

³ For clarity, we will sometimes write $\widehat{f}(n)$ as \widehat{f}_n .

4.1. Energy estimate

Given a smooth solution v to (3.1), let w be as in (3.3). Then from (3.4), we have

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{H^s}^2 &= \frac{d}{dt} \|w(t)\|_{H^s}^2 \\ &= -2 \operatorname{Re} i \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \langle n \rangle^{2s} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} \\ &= \frac{1}{2} \operatorname{Re} i \sum_{\Gamma(\bar{n})} \psi_s(\bar{n}) e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}, \end{aligned} \quad (4.1)$$

where $\bar{n} = (n_1, n_2, n_3, n)$,

$$\Gamma(\bar{n}) := \{(n_1, n_2, n_3, n) \in \mathbb{Z}^4 : n_1 - n_2 + n_3 = n \text{ and } n_1, n_3 \neq n\}$$

and

$$\psi_s(\bar{n}) = \langle n_1 \rangle^{2s} - \langle n_2 \rangle^{2s} + \langle n_3 \rangle^{2s} - \langle n \rangle^{2s}. \quad (4.2)$$

The second equality in (4.1) follows by a symmetrisation argument. Indeed, a relabelling of the sum implies

$$\operatorname{Re} i \sum_{\Gamma(\bar{n})} \langle n \rangle^{2s} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} = \operatorname{Re} i \sum_{\Gamma(\bar{n})} \langle n_2 \rangle^{2s} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}.$$

Using the fact that $\operatorname{Re} ia = -\operatorname{Re} i\bar{a}$ for all $a \in \mathbb{C}$, a relabelling also shows

$$\operatorname{Re} i \sum_{\Gamma(\bar{n})} \langle n \rangle^{2s} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} = -\operatorname{Re} i \sum_{\Gamma(\bar{n})} \langle n_1 \rangle^{2s} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}.$$

This symmetrization puts us in a position to apply Lemma 2.2 later. Writing

$$\frac{d}{dt} \left(\frac{e^{it\phi(\bar{n})}}{i\phi(\bar{n})} \right) = e^{it\phi(\bar{n})},$$

and applying the product rule in reverse, (4.1) implies

$$\begin{aligned} \frac{d}{dt} \|w(t)\|_{H^s}^2 &= \frac{1}{2} \operatorname{Re} \frac{d}{dt} \left[\sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n} \right] \\ &\quad - \frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} \partial_t (w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}). \end{aligned} \quad (4.3)$$

We now define the modified energy:

$$\begin{aligned} E_{s,t}(z) &= \|z\|_{H^s}^2 - \frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} z_{n_1} \overline{z_{n_2}} z_{n_3} \overline{z_n} \\ &=: \|z\|_{H^s}^2 + R_{s,t}(z). \end{aligned}$$

Then, it follows from (4.3) that for any solution w to (3.4), we have

$$\frac{d}{dt} E_{s,t}(w) = -\frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} \partial_t (w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}). \quad (4.4)$$

At first glance it seems like the modified energy (4.5) is non-autonomous in time. However, this time dependence is only superficial. Writing the modified energy in terms of $y := S(t)z$, we have

$$E_s(y) := E_{s,t}(S(-t)y) = \|y\|_{H^s}^2 + R_s(y), \quad (4.5)$$

where

$$R_s(y) := -\frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} y_{n_1} \overline{y_{n_2}} y_{n_3} \overline{y_n}. \quad (4.6)$$

Now, the nonlinear functionals E_s and R_s are clearly autonomous in time.

We now state the following key energy estimate which is of the form (1.14).

Proposition 4.1. *Let (s, α) belong to one of the following regions:*

$$\begin{aligned} \text{(i)} \quad & s > 1, \quad \text{when} \quad \alpha > \frac{1}{2}, \\ \text{(ii)} \quad & \max\left(\frac{2}{3}, \frac{25}{12} - \alpha\right) < s \leq 1, \quad \text{when} \quad \alpha \geq \frac{5}{4}, \\ \text{(iii)} \quad & \frac{3 - \alpha}{2} < s \leq 1, \quad \text{when} \quad 1 < \alpha < \frac{5}{4}. \end{aligned} \quad (4.7)$$

Then, for sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left| \frac{d}{dt} E_s(\mathbf{P}_{\leq N} v(t)) \right| \leq C \|v(t)\|_{H^{s-\frac{1}{2}-\varepsilon}}^6, \quad (4.8)$$

for all $N \in \mathbb{N}$ and any solution v to (3.2), uniformly in $t \in \mathbb{R}$.

Proof. Using (4.5) and the unitarity of $S(t)$ on $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, it suffices to prove, that for small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left| \frac{d}{dt} E_{s,t}(\mathbf{P}_{\leq N} w(t)) \right| \leq C \|w(t)\|_{H^{s-\frac{1}{2}-\varepsilon}}^6 \quad (4.9)$$

for all $N \in \mathbb{N}$ and any solution w to (3.6), uniformly in $t \in \mathbb{R}$.

Using (4.5), (3.6) and the symmetry between n_1 and n_3 and between n_2 and n in (the appropriate version of) (4.4), we have

$$\frac{d}{dt} E_{s,t}(\mathbf{P}_{\leq N} w) = \mathcal{N}_1(w) + \mathcal{R}_1(w) + \mathcal{N}_2(w) + \mathcal{R}_2(w), \quad (4.10)$$

where

$$\mathcal{N}_1(w) := -\operatorname{Re} i \sum_{\Gamma_N(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} \left(\sum_{\Gamma_N(n_1)} e^{it\phi(\bar{m}, n_1)} w_{m_1} \overline{w_{m_2}} w_{m_3} \right) \overline{w_{n_2}} w_{n_3} \overline{w_n}, \quad (4.11)$$

$$\mathcal{R}_1(w) := -\operatorname{Re} i \sum_{\Gamma_N(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} |w_{n_1}|^2 w_{n_1} \overline{w_{n_2}} w_{n_3} \overline{w_n}, \quad (4.12)$$

$$\mathcal{N}_2(w) := -\operatorname{Re} i \sum_{\Gamma_N(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} w_{n_1} \left(\sum_{\Gamma_N(n_2)} e^{-it\phi(\bar{m}, n_2)} \overline{w_{m_1}} w_{m_2} \overline{w_{m_3}} \right) w_{n_3} \overline{w_n}, \quad (4.13)$$

$$\mathcal{R}_2(w) := -\operatorname{Re} i \sum_{\Gamma_N(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} e^{it\phi(\bar{n})} w_{n_1} |w_{n_2}|^2 \overline{w_{n_2}} w_{n_3} \overline{w_n}. \quad (4.14)$$

Here,

$$\phi(\bar{m}, n_1) := |m_1|^{2\alpha} - |m_2|^{2\alpha} + |m_3|^{2\alpha} - |n_1|^{2\alpha}$$

and

$$\Gamma_N(\bar{n}) := \{(n_1, n_2, n_3, n) \in \Gamma(\bar{n}) : |n_j|, |n| \leq N, \ j = 1, 2, 3\}.$$

From now on, we will simply write $\Gamma(\bar{n})$ instead of $\Gamma_N(\bar{n})$ as N plays no further role. In the following, we heavily make use of the Fourier lattice property of $H^s(\mathbb{T})$; namely, that the H^s -norm depends only on the absolute value of the Fourier coefficients. In particular, we assume all the Fourier coefficients w_n are real and non-negative. Moreover, as we never make use of the oscillatory factors such as $e^{it\phi(\bar{n})}$, we will neglect explicitly writing them.

Consider the first scenario (i). For $s > 1$, Lemma 2.3 and Lemma 2.2 imply we have

$$\frac{|\psi_s(\bar{n})|}{|\phi(\bar{n})|} \leq \langle n_{\max} \rangle^{2s-2\alpha}. \quad (4.15)$$

We first estimate $\mathcal{N}_1(w)$ by decomposing the sum into two cases depending on which frequency attains n_{\max} .

• **Case 1:** $n_{\max} = |n_1|$

From the conditions $n = n_1 - n_2 + n_3$ and $n_1 = m_1 - m_2 + m_3$ we have $\max(|n_2|, |n_3|, |n|) \gtrsim |n_1|$ and $\max_{j=1,2,3} |m_j| \gtrsim |n_1|$, respectively. We assume $|n_2| \gtrsim |n_1|$ and $|m_1| \gtrsim |n_1|$ as the other cases are similar. Hence, we have

$$\langle n_{\max} \rangle^{2s-2\alpha} \lesssim \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_{\max} \rangle^{-2\alpha+1+2\varepsilon} \lesssim \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon}. \quad (4.16)$$

Using (4.15), (4.16) and Young's inequality for the convolution of sequences we have

$$\begin{aligned} |\mathcal{N}_1(w)| &\lesssim \sum_{\Gamma(\bar{n})} \left(\sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right) \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} w_{n_3} w_n \\ &\lesssim \left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^2} \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|_{\ell_n^2} \|w_n\|_{\ell_n^1}^2. \end{aligned}$$

A further application of Young's inequality gives

$$\left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^2} \lesssim \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|_{\ell_n^2} \|w_n\|_{\ell_n^1}^2.$$

By Hölder's inequality and choosing ε small enough so that $\frac{1}{2} + \varepsilon \leq s - \frac{1}{2} - \varepsilon$, we have

$$\|w_n\|_{\ell_n^1} \lesssim \|w\|_{H^{\frac{1}{2}+\varepsilon}} \lesssim \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}.$$

Putting this together we get

$$|\mathcal{N}_1(w)| \lesssim \|v\|_{H^{s-\frac{1}{2}-\varepsilon}}^6,$$

which is the desired estimate for $\mathcal{N}_1(w)$.

• **Case 2:** $n_{\max} \in \{|n_2|, |n_3|, |n|\}$

It suffices to assume $n_{\max} = |n_2|$ as the remaining cases follow analogously as below. Similar to Case 1, we have $\max(|n_1|, |n_3|, |n|) \gtrsim |n_2|$. If $|n_1| \gtrsim |n_2|$, we proceed in exactly the same way as Case 1. If instead $|n_3| \gtrsim |n_2|$ or $|n| \gtrsim |n_2|$, say the former as both subcases are similar, we use Young's inequality

$$\begin{aligned} |\mathcal{N}_1(w)| &\lesssim \sum_{\Gamma(\bar{n})} \left(\sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right) \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} \langle n_3 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_3} w_n \\ &\lesssim \left\| \sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^1} \left\| \langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n \right\|_{\ell_n^2}^2 \|w_n\|_{\ell_n^2}. \end{aligned}$$

A further application of Young's inequality and Hölder's inequality gives

$$\left\| \sum_{\Gamma(n_1)} w_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^1} \lesssim \|w_n\|_{\ell_n^1}^3 \lesssim \|w\|_{H^{\frac{1}{2}+\varepsilon}}^3.$$

This completes the case $n_{\max} = |n_2|$ and hence the estimate for $\mathcal{N}_1(w)$. The estimate for $\mathcal{N}_2(w)$ follows from similar arguments. Now we estimate $\mathcal{R}_1(w)$.

• **Case 1:** $n_{\max} = |n_1|$

As before, $\max(|n_2|, |n_3|, |n|) \gtrsim |n_1|$. It suffices to assume $|n_2| \gtrsim |n_1|$, as the subcases $|n_3| \gtrsim |n_1|$ and $|n| \gtrsim |n_1|$ are similar. We have,

$$\begin{aligned} |\mathcal{R}_1(w)| &\lesssim \sum_{\Gamma(\bar{n})} \langle n_1 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_1}^3 \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} w_{n_3} w_n \\ &\lesssim \|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n^3\|_{\ell_n^2} \|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^2}^2. \end{aligned}$$

Using Hölders inequality and then the embedding $\ell_n^\infty \subset \ell_n^2$ we have

$$\|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n^3\|_{\ell_n^2} \lesssim \|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^\infty}^2 \lesssim \|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2} \|w_n\|_{\ell_n^2}^2.$$

Putting everything together, we have shown

$$|\mathcal{R}_1(w)| \lesssim \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}^4 \|w\|_{L^2}^2.$$

• **Case 2:** $n_{\max} \in \{|n_2|, |n_3|, |n|\}$

It suffices to assume $n_{\max} = |n_2|$ as the remaining cases follow analogously as below. As before, $\max(|n_1|, |n_3|, |n|) \gtrsim |n_2|$. We apply the argument of Case 1 if $|n_1| \gtrsim |n_2|$. Instead, if $|n_3| \gtrsim |n_2|$, the remaining case being similar, Young's inequality, the embedding $\ell_n^2 \subset \ell_n^3$ and Hölder's inequality yield

$$\begin{aligned} |\mathcal{R}_1(w)| &\lesssim \sum_{\Gamma(\bar{n})} w_{n_1}^3 \langle n_2 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_2} \langle n_3 \rangle^{s-\frac{1}{2}-\varepsilon} w_{n_3} w_n \\ &\lesssim \|w_n\|_{\ell_n^3}^3 \|\langle n \rangle^{s-\frac{1}{2}-\varepsilon} w_n\|_{\ell_n^2}^2 \|w_n\|_{\ell_n^1} \\ &\lesssim \|w\|_{L^2}^3 \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}^3. \end{aligned}$$

This completes the estimate for $\mathcal{R}_1(w)$ and the estimate for $\mathcal{R}_2(w)$ is similar. Thus, we have established (4.8) in the region (i).

We now move onto establishing (4.8) when $\frac{1}{2} < s \leq 1$. This regime is responsible for the regions (ii) and (iii) in (4.7). As before, we begin with $\mathcal{N}_1(w)$. Notice that since $s \leq 1$, we can no longer apply Lemma 2.2. We set $\sigma = s - \frac{1}{2} - \varepsilon$ and define $\tilde{w}_n = \langle n \rangle^\sigma w_n$. Without loss of generality, we suppose $|n_1| \lesssim |m_1|$. The regularity restriction of $s > \frac{2}{3}$ arises from the following estimate:

$$\left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \right\|_{\ell_{n_1}^\infty} \lesssim \|w\|_{H^{\frac{1}{6}}}^2 \|w\|_{H^\sigma} \lesssim \|w\|_{H^\sigma}^3, \quad (4.17)$$

where the second inequality holds provided $s > \frac{2}{3}$. We decompose the sum in $\mathcal{N}_1(w)$ into a few cases depending on which frequency attains n_{\max} .

• **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

In this case, it is clear from (1.16) and Lemma 2.3 that

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_1| \quad (4.18)$$

and from (4.2) and the mean value theorem,

$$|\psi_s(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_1|. \quad (4.19)$$

Hence with (4.17), we have

$$\begin{aligned} \text{RHS of (4.11)} &\lesssim \sum_{\Gamma(\bar{n})} \frac{n_{\max}^{2s-1}}{n_{\max}^{2\alpha-1}} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{\langle n \rangle^\sigma \langle n_1 \rangle^{\sigma-\frac{1}{6}} \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma} \left\| \sum_{\Gamma(n_1)} \langle m_1 \rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \right\|_{\ell_{n_1}^\infty} \\ &\lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\bar{n})} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{n_{\max}^\nu \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma}, \end{aligned}$$

where

$$\nu = 2\alpha - 2s + 2\sigma - \frac{1}{6} = 2\alpha - \frac{7}{6} - 2\varepsilon > 0.$$

By the Cauchy-Schwarz inequality, we bound this by

$$\|w\|_{H^\sigma}^6 \left(\sum_{\Gamma(\bar{n})} \frac{1}{n_{\max}^{2\nu} \langle n_2 \rangle^{2\sigma} \langle n_3 \rangle^{2\sigma}} \right)^{\frac{1}{2}}$$

and we are done, provided we show

$$\sum_{\Gamma(\bar{n})} \frac{1}{n_{\max}^{2\nu} \langle n_2 \rangle^{2\sigma} \langle n_3 \rangle^{2\sigma}} \lesssim 1. \quad (4.20)$$

For $\delta > 0$ sufficiently small, we have

$$\sum_{\Gamma(\bar{n})} \frac{1}{n_{\max}^{2\nu} \langle n_2 \rangle^{2\sigma} \langle n_3 \rangle^{2\sigma}} \sim \sum_{\substack{n_1, n_2, n_3 \\ |n_2|, |n_3| \ll |n_1|}} \frac{1}{\langle n_1 \rangle^{1+\delta} \langle n_2 \rangle^{1+\delta} \langle n_3 \rangle^{1+\delta}} \frac{\langle n_2 \rangle^{1+\delta-2\sigma} \langle n_3 \rangle^{1+\delta-2\sigma}}{\langle n_1 \rangle^{2\nu-1-\delta}}.$$

Thus, provided

$$2\nu > 1 \quad \text{and} \quad 4\sigma + 2\nu > 3, \quad (4.21)$$

we have

$$\frac{\langle n_2 \rangle^{1+\delta-2\sigma} \langle n_3 \rangle^{1+\delta-2\sigma}}{\langle n_1 \rangle^{2\nu-1-\delta}} \lesssim \frac{1}{\langle n_1 \rangle^{2\nu+4\sigma-3-3\delta}} \lesssim 1,$$

and hence (4.20) follows. The first condition in (4.21) requires $\alpha > \frac{5}{6}$, while the last condition requires $s > \frac{11}{6} - \alpha$.

• **Case 2:** $|n| \sim |n_2| \gg |n_1|, |n_3|$

We have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha}, \quad (4.22)$$

and using (4.17) leads to

$$\text{RHS of (4.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\bar{n})} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{n_{\max}^\nu \langle n_1 \rangle^{\sigma-\frac{1}{6}} \langle n_3 \rangle^\sigma},$$

where

$$\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon > 0.$$

Using Cauchy-Schwarz as in the previous case, we sum over n_1, n_3 and n_2 provided

$$2\nu > 1 \quad \text{and} \quad 2\nu + 4\sigma - \frac{1}{3} > 3$$

and hence $s > \frac{11}{6} - \alpha$. Notice the first condition above requires $\alpha > \frac{3}{4}$.

• **Case 3:** $|n| \sim |n_3| \gg |n_1|, |n_2|$

We have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_3| \quad \text{and} \quad |\psi_s(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_3|.$$

Thus

$$\text{RHS of (4.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\bar{n})} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{n_{\max}^\nu \langle n_1 \rangle^{\sigma-\frac{1}{6}} \langle n_2 \rangle^\sigma},$$

where

$$\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon > 0.$$

Using Cauchy-Schwarz as in the previous cases, we sum over n_1, n_2 and n_3 provided

$$2\nu > 1 \quad \text{and} \quad 2\nu + 4\sigma - \frac{1}{3} > 3,$$

and hence $s > \frac{11}{6} - \alpha$. The first condition above requires $\alpha > \frac{3}{4}$.

• **Case 4:** $|n_1| \sim |n_2| \gg |n_3|, |n|$

We have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_3| \quad \text{and} \quad |\psi_s(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_3|$$

and we proceed as in Case 1 as long as $s > \frac{11}{6} - \alpha$ and $\alpha > \frac{5}{6}$.

• **Case 5:** $|n_1| \sim |n_3| \gg |n_2|, |n|$

We have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha},$$

and hence

$$\text{RHS of (4.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\bar{n})} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{n_{\max}^\nu \langle n \rangle^\sigma \langle n_2 \rangle^\sigma},$$

where

$$\nu = 2\alpha - 2s + 2\sigma - \frac{1}{6} = 2\alpha - \frac{7}{6} - 2\varepsilon > 0.$$

By Cauchy-Schwarz and summing in n, n_2 and n_1 as long as

$$2\nu > 1 \quad \text{and} \quad 2\nu + 4\sigma > 3,$$

and hence $s > \frac{11}{6} - \alpha$. The first condition above is satisfied provided $\alpha > \frac{5}{6}$.

• **Case 6:** $|n_2| \sim |n_3| \gg |n_1|, |n|$

We have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_1| \quad \text{and} \quad |\psi_s(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_1|,$$

and we proceed as in Case 4 as long as $s > \frac{11}{6} - \alpha$ and $\alpha > \frac{3}{4}$.

• **Case 7:** $|n_1| \sim |n_2| \sim |n_3| \gg |n|$

From Lemma 2.3, we have $|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha}$ and hence

$$\text{RHS of (4.11)} \lesssim \|w\|_{H^\sigma}^3 \sum_{\Gamma(\bar{n})} \frac{n_{\max}^{2s} \tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{n_{\max}^{2\alpha} n_{\max}^{3\sigma-\frac{1}{6}} \langle n \rangle^\sigma}.$$

Applying Cauchy-Schwarz in n_2, n_3 and n , we sum provided

$$2\left(2\alpha + 3\sigma - \frac{1}{6} - 2s\right) > 2 \quad \text{and} \quad 2\alpha - 2s + 3\sigma - \frac{1}{6} + \sigma > \frac{3}{2},$$

which requires $\alpha + s > \frac{11}{6}$ and $2\alpha + s > \frac{8}{3}$. Notice that when $\alpha > \frac{5}{6}$, this latter condition is superseded by the former. The remaining cases of the form $|n_{j_1}| \sim |n_{j_2}| \sim |n_{j_3}| \gg |n_{j_4}|$ with distinct $j_k \in \{1, 2, 3, 4\}$ ($n_{j_4} := n$) are similar and are thus omitted.

• **Case 8:** $|n| \sim |n_1| \sim |n_2| \sim |n_3|$

We distinguish when α is ‘close too’ or ‘far from’ 1.

• **Subcase 8.1:** $1 < \alpha < \frac{5}{4}$

By Lemma 2.3, we have

$$\begin{aligned} & \sum_{\Gamma(\bar{n})} \frac{n_{\max}^{2s}}{|n - n_1| |n - n_3| n_{\max}^{2\alpha-2} \langle n \rangle^\sigma} \left(\prod_{j=1}^3 \frac{1}{\langle n_j \rangle^\sigma} \right) \left(\sum_{\Gamma(n_1)} \tilde{w}_{m_1} w_{m_2} w_{m_3} \right) \tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n \\ & \lesssim \sum_{\Gamma(\bar{n})} \frac{1}{|n - n_1| |n - n_3| n_{\max}^\nu} \left(\sum_{\Gamma(n_1)} \tilde{w}_{m_1} w_{m_2} w_{m_3} \right) \tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n, \end{aligned}$$

where

$$\nu = 2\alpha - 2 - 2s + 4\sigma = 2(\alpha + s - 2 - 2\varepsilon) > 0, \quad (4.23)$$

provided

$$\alpha + s > 2. \quad (4.24)$$

By Cauchy-Schwarz in n, n_1 and n_3 , followed by summing in n and n_3 , we get

$$\begin{aligned} & \lesssim \|w\|_{H^\sigma}^3 \left\| \frac{1}{\langle n - n_1 \rangle \langle n - n_3 \rangle \langle n_{\max} \rangle^\nu} \left(\sum_{\Gamma(n_1)} \tilde{w}_{m_1} w_{m_2} w_{m_3} \right) \right\|_{\ell_{n, n_1, n_3}^2} \\ & \lesssim \|w\|_{H^\sigma}^3 \left\| \frac{1}{\langle n_1 \rangle^\nu} \sum_{\Gamma(n_1)} \tilde{w}_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^2}. \end{aligned}$$

Imposing

$$\alpha + s < 2 + \frac{1}{4} \quad (4.25)$$

implies $\nu < \frac{1}{2}$ so that we can apply Hölder’s inequality and then Young’s inequality, with exponents⁴

$$\frac{1 - 2\nu +}{2} + 2 = \frac{1}{2} + \frac{2}{2 - \nu +},$$

to obtain

$$\begin{aligned} & \lesssim \|w\|_{H^\sigma}^3 \left\| \sum_{\Gamma(n_1)} \tilde{w}_{m_1} w_{m_2} w_{m_3} \right\|_{\ell_{n_1}^{\frac{2}{1-2\nu+}}} \\ & \lesssim \|w\|_{H^\sigma}^4 \|w_n\|_{\ell_n^{\frac{2}{2-\nu+}}}^2. \end{aligned}$$

Once more with Hölder’s inequality, we have

$$\lesssim \|w\|_{H^\sigma}^6 \|\langle n \rangle^{-\sigma}\|_{\ell_n^{\frac{2}{1-\nu+}}}^2 \lesssim \|w\|_{H^\sigma}^6,$$

provided

⁴ Here, we use the notation $a -$ (respectively, $a +$) to denote $a - \delta$ (respectively, $a + \delta$), where $0 < \delta \ll 1$ is extremely small.

$$\frac{2\sigma}{1-\nu+} > 1.$$

Using (4.23), this last condition requires

$$s > \frac{3-\alpha}{2}. \quad (4.26)$$

Putting the conditions (4.24), (4.25) and (4.26) together implies we must enforce in this subcase

$$\max\left(\frac{1}{2}, \frac{3-\alpha}{2}, 2-\alpha\right) < s \leq \min\left(1, \frac{9}{4}-\alpha\right),$$

where the upper bound is strict if the minimum is $\frac{9}{4}-\alpha$. Now as $\alpha \leq \frac{5}{4}$, $\min(1, \frac{9}{4}-\alpha) = 1$ and this implies the range

$$\frac{3-\alpha}{2} < s \leq 1.$$

• **Subcase 8.2:** $\alpha \geq \frac{5}{4}$

Given $n \in \mathbb{Z}$, let

$$\Gamma(n, \rho) = \Gamma(n) \cap \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : \rho = (n - n_1)(n - n_3) \in \mathbb{Z}\}.$$

From (2.1) we have, for any $\delta > 0$, there exists a $C_\delta > 0$ such that

$$|\#\Gamma(n, \rho)| \lesssim C_\delta |\rho|^\delta. \quad (4.27)$$

By Lemma 2.3 and (4.17), we have

$$\begin{aligned} |\mathcal{N}_1(w)| &\lesssim \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{n_{\max}^{2s}}{|\rho| n_{\max}^{2\alpha-2}} \frac{\tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n}{\langle n \rangle^\sigma \langle n_1 \rangle^{\sigma-\frac{1}{6}} \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma} \left\| \sum_{\Gamma(n)} \langle m_1 \rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \right\|_{\ell_{n_1}^\infty} \\ &\lesssim \|w\|_{H^\sigma}^3 \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{1}{|\rho| n_{\max}^\nu} \tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n, \end{aligned}$$

where

$$\nu = 2\alpha - 2 - 2s + 4\sigma - \frac{1}{6} = 2\alpha + 2s - \frac{25}{6} - 2\varepsilon,$$

which is positive provided

$$\alpha + s > \frac{25}{12}. \quad (4.28)$$

To continue, we follow the argument in [33, Proposition 6.1] and for completeness we detail it here. By Cauchy-Schwarz, (4.27) and Lemma 2.4, we have

$$\begin{aligned} &\lesssim \|w\|_{H^\sigma}^4 \left[\sum_n \left(\sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{1}{|\rho| n_{\max}^\nu} \tilde{w}_{n_2} \tilde{w}_{n_3} \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \|w\|_{H^\sigma}^4 \left[\sum_n \left(\sum_{\rho \neq 0} \frac{1}{|\rho|^{1+2\delta}} \sum_{\Gamma(n, \rho)} 1 \right) \sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{1}{|\rho|^{1-2\delta} n_{\max}^{2\nu}} \tilde{w}_{n_2}^2 \tilde{w}_{n_3}^2 \right]^{\frac{1}{2}} \\ &\lesssim \|w\|_{H^\sigma}^4 \left(\sum_{n_2, n_3} \tilde{w}_{n_2}^2 \tilde{w}_{n_3}^2 \sum_{n_1 \neq n_2} \frac{1}{|n_1 - n_2|^{1-2\delta} \langle n_1 \rangle^{2\nu}} \right)^{\frac{1}{2}} \\ &\lesssim \|w\|_{H^\sigma}^6, \end{aligned}$$

where from (4.28) we choose $\delta > 0$ small enough so that $\delta < \nu$.

The required range for s in this subcase is

$$\max\left(\frac{2}{3}, \frac{25}{12} - \alpha\right) < s \leq 1.$$

This completes the estimates for $\mathcal{N}_1(w)$. The estimate for $\mathcal{N}_2(w)$ follows analogously.

We now move onto bounding $\mathcal{R}_1(w)$. Writing

$$m(\bar{n}) := \frac{|\psi_s(\bar{n})|}{|\phi(\bar{n})| \langle n_1 \rangle^{3\sigma} \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma \langle n \rangle^\sigma},$$

it suffices to show

$$\sum_{\Gamma(\bar{n})} m(\bar{n}) |\tilde{w}_{n_1}|^3 \tilde{w}_{n_2} \tilde{w}_{n_3} \tilde{w}_n \lesssim \|\tilde{w}\|_{L^2}^6. \quad (4.29)$$

As above, we divide into a few cases.

• **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

Using (4.18) and (4.19) we have

$$m(\bar{n}) \lesssim \frac{1}{n_{\max}^\nu \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma},$$

where $\nu = 2\alpha - 2s + 4\sigma = 2\alpha + 2s - 2 - 4\varepsilon$. By Cauchy-Schwarz in n_2, n_3 and n and the embedding $\ell_n^2 \subset \ell_n^6$, we have

$$\text{LHS of (4.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \left(\sum_n \tilde{w}_n^2 \sum_{n_2, n_3} \frac{1}{n_{\max}^{2\nu} \langle n_2 \rangle^{2\sigma} \langle n_3 \rangle^{2\sigma}} \right)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6,$$

where we can sum provided $4\sigma + 2\nu > 2$ which requires $s > 1 - \frac{1}{2}\alpha$.

• **Case 2:** $|n| \sim |n_3| \gg |n_1|, |n_2|$

Using (4.22), we have

$$m(\bar{n}) \lesssim \frac{1}{n_{\max}^\nu \langle n_1 \rangle^{3\sigma} \langle n_2 \rangle^\sigma},$$

where $\nu = 2\alpha - 2s + 2\sigma = 2\alpha - 1 - 2\varepsilon$. By Cauchy-Schwarz in n_1, n_2 and n_3 and the embedding $\ell_n^2 \subset \ell_n^6$, we have

$$\text{LHS of (4.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \left(\sum_{n_3} \tilde{w}_{n_3}^2 \sum_{n_1, n_2} \frac{1}{n_{\max}^{2\nu} \langle n_1 \rangle^{6\sigma} \langle n_2 \rangle^{2\sigma}} \right)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6,$$

where we can sum provided $s > \max(\frac{2}{3}, 2 - 2\alpha)$.

• **Case 3:** $|n_1| \sim |n_2| \sim |n_3| \sim |n|$

From (1.16), we have

$$m(\bar{n}) \lesssim \frac{n_{\max}^{2s}}{|n - n_1| |n - n_3| n_{\max}^{2\alpha-2}} \frac{1}{\langle n_1 \rangle^{3\sigma} \langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma \langle n \rangle^\sigma} \sim \frac{1}{|n - n_1| |n - n_3| n_{\max}^\nu},$$

where

$$\nu = 2\alpha - 2 - 2s + 6\sigma > 0,$$

provided $s > \frac{5}{4} - \frac{1}{2}\alpha$. An application of Cauchy-Schwarz then implies

$$\text{LHS of (4.29)} \lesssim \|\tilde{w}\|_{L^2}^5 \left(\sum_n \tilde{w}_n^2 \sum_{n_1, n_3 \neq n} \frac{1}{|n - n_1|^2 |n - n_3|^2} \right)^{\frac{1}{2}} \lesssim \|\tilde{w}\|_{L^2}^6.$$

All remaining cases follow analogously from the methods in either Case 1 or Case 2 above and are thus omitted. This completes the bound for \mathcal{R}_1 . Notice the condition from Case 3 supersedes the conditions from Cases 1 and 2. Furthermore, at least for every $\alpha \geq \frac{7}{6}$, we have

$$\frac{2}{3} \geq \frac{5}{4} - \frac{1}{2}\alpha$$

and hence we obtain the final conditions on s and α of (4.7).

Finally, this completes the proof of (4.9). \square

We also have the following difference estimate for $R_s(v)$. It will be convenient to view R_s as a multi-linear functional

$$R_s(u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}) := \frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} u_{n_1}^{(1)} \overline{u_{n_2}^{(2)}} u_{n_3}^{(3)} \overline{u_n^{(4)}},$$

where $R_s(v, v, v, v) = R_s(v)$.

Proposition 4.2. *Suppose*

$$\begin{aligned} \text{(i)} \quad & s > 1, \quad \text{when} \quad \alpha > \frac{1}{2}, \text{ or} \\ \text{(ii)} \quad & s > \max\left(2 - \alpha, \frac{1}{2}\right), \quad \text{when} \quad \alpha \geq 1. \end{aligned} \tag{4.30}$$

Then, for sufficiently small $\varepsilon > 0$ there exists $C > 0$ such that

$$|R_s(u) - R_s(v)| \leq C \|u - v\|_{H^{s-\frac{1}{2}-\varepsilon}} (\|u\|_{H^{s-\frac{1}{2}-\varepsilon}}^3 + \|v\|_{H^{s-\frac{1}{2}-\varepsilon}}^3)$$

for all $u, v \in H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$.

Proof. By the multi-linearity of $R_s(u)$, it suffices to show

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}, \tag{4.31}$$

where $\sigma := s - \frac{1}{2} - \varepsilon$.

We first consider case (i) in (4.30). Using (4.6) and Lemmas 2.3 and 2.2 we have

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \sum_{\Gamma(\bar{n})} \langle n_{\max} \rangle^{2s-2\alpha} |u_{n_1}^{(1)}| |u_{n_2}^{(2)}| |u_{n_3}^{(3)}| |u_n^{(4)}|.$$

Similar to the proof of (4.1), it suffices to consider the following case when $n_{\max} = |n_1|$ and $|n_2| \gtrsim |n_1|$. Using Young's and Hölder's inequality, we get

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \|\langle n_1 \rangle^\sigma u_{n_1}^{(1)}\|_{\ell_{n_1}^2} \|\langle n_2 \rangle^\sigma u_{n_2}^{(2)}\|_{\ell_{n_2}^2} \|u_{n_3}^{(3)}\|_{\ell_{n_3}^1} \|u_n^{(4)}\|_{\ell_n^1} \lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}.$$

We now consider case (ii) in (4.30). As it is already contained within the case $s > 1$ and $\alpha > \frac{1}{2}$ proved above, we now bound $|R_s(\{u^{(j)}\}_{j=1}^4)|$ when $s \leq 1$ for $\alpha \geq 1$. Given such an s , let $\sigma = s - \frac{1}{2} - \varepsilon$. We have

$$|R_s(\{u^{(j)}\}_{j=1}^4)| \lesssim \sum_{\Gamma(\bar{n})} m(\bar{n}) |\tilde{u}_{n_1}^{(1)}| |\tilde{u}_{n_2}^{(2)}| |\tilde{u}_{n_3}^{(3)}| |\tilde{u}_n^{(4)}|, \tag{4.32}$$

where

$$m(\bar{n}) = \frac{|\psi_s(\bar{n})|}{|\phi(\bar{n})| \langle n \rangle^\sigma} \prod_{j=1}^3 \frac{1}{\langle n_j \rangle^\sigma}.$$

As in the proof of Proposition 4.1, we consider a few cases depending on n_{\max} .

• **Case 1:** $|n| \sim |n_1| \gg |n_2|, |n_3|$

In this case, we have

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_1| \quad \text{and} \quad |\psi(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_1|,$$

and thus

$$m(\bar{n}) \lesssim \frac{1}{n_{\max}^\nu} \frac{1}{\langle n_2 \rangle^\sigma \langle n_3 \rangle^\sigma},$$

where $\nu = 2\alpha - 1 - 2\varepsilon > 0$. By Cauchy-Schwarz inequality,

$$\text{RHS of (4.32)} \lesssim \prod_{j=1}^3 \|u^{(j)}\|_{H^\sigma}^3 \left(\sum_{n, n_2, n_3 \in \mathbb{Z}} \frac{1}{n_{\max}^{2\nu} \langle n_2 \rangle^{2\sigma} \langle n_3 \rangle^{2\sigma}} (\tilde{u}_n^{(4)})^2 \right)^{\frac{1}{2}} \lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}^4,$$

where we can sum in n_2 and n_3 provided $\nu + 2\sigma > 1$, which requires

$$\alpha + s > \frac{3}{2}.$$

• **Case 2:** $|n| \sim |n_2| \gg |n_1|, |n_3|$

Here we use $|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha}$ which implies

$$m(\bar{n}) \lesssim \frac{1}{n_{\max}^\nu \langle n_1 \rangle^\sigma \langle n_3 \rangle^\sigma},$$

where $\nu = 2\alpha - 1 - 2\varepsilon > 0$. We proceed as in Case 1 by using Cauchy-Schwarz and summing in n_1 and n_3 with $\tilde{v}_{n_2}^2$ absorbing the remaining n_2 summation.

It is easy to check that all remaining Cases 3 through 7 as explicated in the proof of Proposition 4.1 follow analogously to the two cases above.

• **Case 3:** $|n| \sim |n_1| \sim |n_2| \sim |n_3|$

We can only use the lower bound of Lemma 2.3 and this implies

$$m(\bar{n}) \lesssim \frac{1}{|n - n_1| |n - n_3| n_{\max}^\nu},$$

where $\nu = 2\alpha - 2 - 2s + 4\sigma$ which is non-negative provided

$$\alpha + s > 2.$$

Then Cauchy-Schwarz over $\Gamma(\bar{n})$ gives

$$\begin{aligned} \text{RHS of (4.32)} &\lesssim \prod_{j=1}^3 \|u^{(j)}\|_{H^\sigma}^3 \left(\sum_{n \in \mathbb{Z}} (\tilde{u}_n^{(4)})^2 \sum_{n_1, n_3 \in \mathbb{Z}} \frac{1}{|n - n_1|^2 |n - n_3|^2} \right)^{\frac{1}{2}} \\ &\lesssim \prod_{j=1}^4 \|u^{(j)}\|_{H^\sigma}^4. \end{aligned}$$

This completes the proof of (4.31). \square

Remark 4.3. Since $2 - \alpha \leq \frac{25}{12} - \alpha$ and $2 - \alpha \leq \frac{3-\alpha}{2}$ for $\alpha \geq 1$, the restriction (4.7) for the energy estimate supersedes (4.30) (ii), which is the condition for the correction term $R_s(u)$.

4.2. Proof of Theorem 1.2 (ii)

In this subsection, we follow the argument introduced in [38] to conclude quasi-invariance of Gaussian measures μ_s for $\alpha > \frac{1}{2}$ and those s given in Proposition 4.1. In particular, we conclude Theorem 1.2 (ii) and the $s > 1$ portion of Theorem 1.3.

We define the following measures:

$$d\rho_s = F_s(u) d\mu_s \quad \text{and} \quad d\rho_{s,N} = F_{s,N}(u) d\mu_s,$$

where

$$F_s(u) = \exp\left(-\frac{1}{2}E_s(u) + \frac{1}{2}\|u\|_{H^s}^2\right) = \exp\left(-\frac{1}{2}R_s(u)\right) \quad \text{and} \quad F_{s,N}(u) = F_s(\mathbf{P}_{\leq N}u).$$

The measure $\rho_{s,N}$ can also be expressed as

$$d\rho_{s,N} = Z_{s,N}^{-1} \exp\left(-\frac{1}{2}E_s(\mathbf{P}_{\leq N}u)\right) du_{\leq N} \times d\mu_{s,N}^{\perp}, \quad (4.33)$$

where $du_{\leq N}$ denotes the Lebesgue measure on \mathbb{C}^{2N+1} . The constant $Z_{s,N}^{-1}$ is the normalisation constant associated to the measure $\mu_{s,N}$ which is given by

$$d\mu_{s,N} = Z_{s,N}^{-1} e^{-\frac{1}{2}\|\mathbf{P}_{\leq N}u\|_{H^s}^2} du_N.$$

In particular, $\mu_{s,N}$ is the probability measure induced under the map

$$\omega \in \Omega \mapsto u_{\leq N}^{\omega}(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}.$$

Likewise, $\mu_{s,N}^{\perp}$ is the probability measure induced under the map

$$\omega \in \Omega \mapsto u_{>N}^{\omega}(x) = \sum_{|n| > N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{inx}.$$

Note that we do not require F_s and $F_{s,N}$ to be integrable with respect to μ_s . Hence ρ_s and $\rho_{s,N}$ are not necessarily probability measures. However, as the quasi-invariance argument is purely local (see the proof of Theorem 1.2 (i) below), it suffices to have $F_{s,N} \in L_{\text{loc}}^1(\mu_s)$ and with convergence to F_s . This is the content of the next proposition, whose proof can be found in [38, Proposition 2.1].

Proposition 4.4. *Let s be as in (4.30). Then, for every bounded set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, we have*

$$\lim_{N \rightarrow \infty} \int_A |F_{s,N}(u) - F_s(u)| d\mu_s(u) = 0$$

and in particular

$$\lim_{N \rightarrow \infty} |\rho_{s,N}(A) - \rho_s(A)| = 0$$

The next result states important properties of the truncated flow Φ_N .

Proposition 4.5. *Let s be as in Proposition 1.1 be such that the flow Φ of FNLS (1.1) is globally well-defined. Then, the following statements hold:*

(i) *For every $R > 0$ and $T > 0$, there exists $C(R, T) > 0$ such that*

$$\Phi_N(t)(B_R) \subset B_{C(R,T)}$$

for all $t \in [0, T]$ and for all $N \in \mathbb{N} \cup \{\infty\}$. Here, $\Phi_{\infty} := \Phi$ denotes the untruncated flow.

(ii) Let $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ be a compact set and $t \in \mathbb{R}$. Then, for every $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\|\Phi(t)(u) - \Phi_N(t)(u)\|_{H^s} < \delta,$$

for any $u \in A$ and any $N \geq N_0$. Furthermore, we have

$$\Phi(t)(A) \subset \Phi_N(t)(A + B_\delta)$$

for all $N \geq N_0$.

We also have the following local-in-time version of Proposition 4.5.

Proposition 4.6. Let s be as in Proposition 1.1 (ii) be such that the flow Φ of FNLS (1.1) is only locally well-defined. Then, the following statements hold:

(i) Then, for every $R > 0$, there exist $T(R) > 0$ and $C(R) > 0$ such that

$$\Phi_N(t)(B_R) \subset B_{C(R)}$$

for all $t \in [0, T(R)]$ and for all $N \in \mathbb{N} \cup \{\infty\}$.

(ii) Let $A \subset B_R \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$ be a compact set and denote by $T(R) > 0$ the local existence time of the solution map Φ defined on B_R . Then, for every $\delta > 0$, there exists $N_0 \in \mathbb{N}$, such that

$$\|\Phi(t)(u) - \Phi_N(t)(u)\|_{H^s} < \delta,$$

for any $u \in A$, $N \geq N_0$ and $t \in [0, T(R)]$. Furthermore, we have

$$\Phi(t)(A) \subset \Phi_N(t)(A + B_\delta)$$

for all $t \in [0, T(R)]$ and for all $N \geq N_0$.

The proof of Proposition 4.5 (i) follows from the global well-posedness of FNLS (1.1) when $\alpha > \frac{10\alpha+1}{12}$, while the proof of Proposition 4.6 (i) follows by the local existence theory (for short times) when $\frac{1}{2} < \alpha \leq \frac{10\alpha+1}{12}$. The proof of Proposition 4.5 (ii) follows from the arguments in [33, Appendix B] using the existence theory in Appendix B, [1] and [8].

Proof of Theorem 1.2 (i). In the following we fix s and α satisfying the conditions of Propositions 4.1 and 4.2. As long as the conclusions of these propositions are satisfied, the following general argument due to [38] implies the quasi-invariance of μ_s (either globally or locally in time). For clarity, we will only detail the following arguments in the case when FNLS (1.1) admits a globally well-defined flow Φ (see Proposition 1.1). We obtain local-in-time quasi-invariance from the same arguments by suitably restricting to the local well-posedness lifetime where necessary.

Given $t > 0$, by the inner regularity of the measure μ_s , it is enough to show that

$$A \subset H^{s-\frac{1}{2}-\varepsilon} \text{ compact and } \mu_s(A) = 0 \implies \mu_s(\Phi(-t)A) = 0. \quad (4.34)$$

From Proposition 4.2 with $u = 0$, we have $0 < \exp(R_s(v)) < \infty$ for almost all $v \in A$. Hence the implication (4.34) is equivalent to the following implication:

$$A \subset H^{s-\frac{1}{2}-\varepsilon} \text{ compact and } \rho_s(A) = 0 \implies \rho_s(\Phi(-t)A) = 0.$$

As A is compact, there exists $R > 0$ such that $A \subset B_R$. Then, by Proposition 4.5, there exists a constant $C(R) > 0$ such that

$$\Phi(\tau)(B_{2R}) \cup \Phi_N(\tau)(B_{2R}) \subset B_{C(R)} \quad (4.35)$$

for all $\tau \in [0, t]$. For a measurable $D \subset B_{2R}$, it follows from (4.33), Liouville's theorem and the invariance of complex Gaussians under rotations, that

$$\begin{aligned} \left| \frac{d}{d\tau} \rho_{s,N}(\Phi_N(\tau)(D)) \right| &= \left| \frac{d}{d\tau} Z_{s,N}^{-1} \int_{\Phi_N(\tau)(D)} \exp\left(-\frac{1}{2} E_s(\mathbf{P}_{\leq N} u)\right) du_{\leq N} \times d\mu_{s,N}^{\perp} \right| \\ &= \left| Z_{s,N}^{-1} \int_D \frac{d}{d\tau} \exp\left(-\frac{1}{2} E_s(\Phi(\tau)(\mathbf{P}_{\leq N} u))\right) du_{\leq N} \times d\mu_{s,N}^{\perp} \right|. \end{aligned}$$

Using the energy estimate of Proposition 4.1 along with (4.35) we have

$$\left| \frac{d}{d\tau} \exp\left(-\frac{1}{2} E_s(\Phi(\tau)(\mathbf{P}_{\leq N} u))\right) \right| \leq C(R) \exp\left(-\frac{1}{2} E_s(\Phi(\tau)(\mathbf{P}_{\leq N} u))\right)$$

for all $\tau \in [0, t]$ and for all $u \in D$. Combining the above we have

$$\begin{aligned} \left| \frac{d}{d\tau} \rho_{s,N}(\Phi_N(\tau)(D)) \right| &\leq Z_{s,N}^{-1} C(R) \int_D \frac{d}{d\tau} \exp\left(-\frac{1}{2} E_s(\Phi(\tau)(\mathbf{P}_{\leq N} u))\right) du_{\leq N} \times d\mu_{s,N}^{\perp} \\ &= C(R) \rho_{s,N}(\Phi_N(\tau)(D)). \end{aligned}$$

From Gronwall's inequality, we get

$$\rho_{s,N}(\Phi_N(\tau)(D)) \leq e^{C(R)\tau} \rho_{s,N}(D) \quad (4.36)$$

for all $N \in \mathbb{N}$ and for all $\tau \in [0, t]$. By Proposition 4.5 (ii), we have

$$\rho_s(\Phi_N(\tau)(A)) \leq \rho_s(\Phi_N(\tau)(A + B_\delta))$$

for any fixed $\delta > 0$ and N large enough. Further, from Proposition 4.4 for N large enough, we have

$$\rho_s(\Phi_N(\tau)(A + B_\delta)) \leq \rho_{s,N}(\Phi_N(\tau)(A + B_\delta)) + \delta$$

and so

$$\rho_s(\Phi_N(\tau)(A)) \leq \rho_{s,N}(\Phi_N(\tau)(A + B_\delta)) + \delta.$$

Choosing $\delta < R$ so that $A + B_\delta \subset B_{2R}$ and (4.36), can be applied we get

$$\rho_s(\Phi_N(\tau)(A)) \leq e^{C(R)\tau} \rho_{s,N}(A + B_\delta) + \delta.$$

Using Proposition 4.4 to go from $\rho_{s,N}$ back to ρ_s , we have

$$\rho_s(\Phi_N(t)(A)) \leq e^{C(R)t} \rho_s(A + B_\delta) + 2\delta. \quad (4.37)$$

Letting δ approach 0 and using regularity properties of the measure μ_s , we finally obtain

$$\rho_s(\Phi_N(\tau)(A)) \leq e^{C(R)\tau} \lim_{\delta \rightarrow 0} \rho_s(A + B_\delta) = e^{C(R)\tau} \rho_s(A) = 0$$

for any $\tau \in [0, t]$. This completes the proof. \square

5. Improvement for $\alpha > \frac{5}{6}$

In this section, we employ the hybrid argument (Method 4) from [19] in order to lower the regularity threshold we previously obtained using Method 3. Namely, we complete the proofs of Theorem 1.2 (i) and Theorem 1.3 by proving the quasi-invariance of Gaussian measures μ_s under the flow of FNLS (1.1) for regularities satisfying (1.20).

5.1. Alternative energy estimate

Our first port of call is to obtain an energy estimate where we place two factors into the Fourier-Lebesgue space $\mathcal{FL}^{\sigma, \infty}(\mathbb{T})$, where $\sigma < s$. By placing these two factors into this stronger norm, we can lower the regularity restriction; compare (4.7) and (5.1).

Proposition 5.1. *Let $\alpha > \frac{5}{6}$ and*

$$\max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right) < s \leq 1. \quad (5.1)$$

Then, for sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left| \frac{d}{dt} E_s(\mathbf{P}_{\leq N} v(t)) \right|_{t=0} \leq C \|\mathbf{P}_{\leq N} v(0)\|_{\mathcal{FL}^{s-\tilde{\varepsilon}, \infty}}^2 \|\mathbf{P}_{\leq N} v(0)\|_{H^{s-\frac{1}{2}-\varepsilon}}^4, \quad (5.2)$$

for any $N \in \mathbb{N}$, any solution v to (3.2) and for any $0 < \tilde{\varepsilon} < \varepsilon$, uniformly in $t \in \mathbb{R}$.

Proof. Using (4.5), the estimate (5.2) reduces to proving that for small $\varepsilon > 0$, there exists $C > 0$ such that

$$\left| \frac{d}{dt} E_{s,t}(\mathbf{P}_{\leq N} w(t)) \right|_{t=0} \leq C \|\mathbf{P}_{\leq N} w(0)\|_{\mathcal{FL}^{s-\tilde{\varepsilon}, \infty}}^2 \|\mathbf{P}_{\leq N} w(0)\|_{H^{s-\frac{1}{2}-\varepsilon}}^4. \quad (5.3)$$

From (4.10), (4.11), (4.12), (4.13) and (4.14), (5.3) further reduces to showing

$$\left| \sum_{j=1}^2 \mathcal{N}_j(\mathbf{P}_{\leq N} w(0)) + \sum_{j=1}^2 \mathcal{R}_j(\mathbf{P}_{\leq N} w(0)) \right| \leq C \|\mathbf{P}_{\leq N} w(0)\|_{\mathcal{FL}^{s-\tilde{\varepsilon}, \infty}}^2 \|\mathbf{P}_{\leq N} w(0)\|_{H^{s-\frac{1}{2}-\varepsilon}}^4 \quad (5.4)$$

for all $N \in \mathbb{N}$ and uniformly in $t \in \mathbb{R}$. Recalling the decomposition (4.10), we estimate \mathcal{N}_1 and \mathcal{R}_1 , with estimates for \mathcal{N}_2 and \mathcal{R}_2 following analogously. In the following, we simply replace $w(0)$ by w . We consider \mathcal{N}_1 first. Recall that in Cases 1 through 7 of the proof of Proposition 4.1 (ii) and (iii), we obtained

$$|\mathcal{N}_1(w)| \lesssim \|w\|_{H^{s-\frac{1}{2}-\varepsilon}}^6 \quad (5.5)$$

for any $\alpha > \frac{5}{6}$ and for any s satisfying

$$1 \geq s > \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right). \quad (5.6)$$

Then in these cases, we obtain (5.4) by using the embedding (1.19) to put two factors of (5.5) into the required Fourier-Lebesgue space. Note that we could certainly improve upon the regularity lower bound on s in these cases by proving (5.4) ‘directly.’ However, we find that consideration of the remaining case $|n| \sim |n_1| \sim |n_2| \sim |n_3|$ yields a restriction on s given by (5.1). We now describe this remaining case. To simplify notation, we drop the frequency projections $\mathbf{P}_{\leq N}$. Furthermore, we let $\sigma = s - \frac{1}{2} - \varepsilon$ and we set $\tilde{w}_n = \langle n \rangle^\sigma w_n$ and $\mathbf{w}_n = \langle n \rangle^{s-\tilde{\varepsilon}} w_n$. We employ the argument (and the notation) from subcase 8.2 in the proof of Proposition 4.1. We have

$$\begin{aligned} |\mathcal{N}_1(w)| &\lesssim \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{n_{\max}^{2s}}{|\rho| n_{\max}^{2\alpha-2}} \frac{\mathbf{w}_{n_2} \mathbf{w}_{n_3} \tilde{w}_n}{\langle n \rangle^\sigma \langle n_1 \rangle^{\sigma-\frac{1}{6}} \langle n_2 \rangle^{s-\tilde{\varepsilon}} \langle n_3 \rangle^{s-\tilde{\varepsilon}}} \left\| \sum_{\Gamma(n)} \langle m_1 \rangle^{\sigma-\frac{1}{6}} w_{m_1} \overline{w_{m_2}} w_{m_3} \right\|_{\ell_{n_1}^\infty} \\ &\lesssim \|w\|_{H^\sigma}^3 \|w\|_{\mathcal{FL}^{s-\tilde{\varepsilon}, \infty}}^2 \sum_n \sum_{\rho \neq 0} \sum_{\Gamma(n, \rho)} \frac{1}{|\rho| n_{\max}^v} \tilde{w}_n, \end{aligned}$$

where

$$v = 2\alpha + 2s - \frac{19}{6} > 0,$$

provided $\alpha + s > \frac{19}{12}$. Then, by Cauchy-Schwarz and the divisor counting lemma (4.27), we bound the above by

$$\begin{aligned} & \|w\|_{H^\sigma}^4 \|w\|_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}}^2 \left(\sum_n \sum_{\Gamma(n)} \frac{1}{|n-n_1|^{1-2\delta} |n-n_3|^{1-2\delta} n_{\max}^{2\nu}} \right)^{\frac{1}{2}} \\ & \lesssim \|w\|_{H^\sigma}^4 \|w\|_{\mathcal{F}L^{s-\tilde{\varepsilon},\infty}}^2 \left(\sum_n \frac{1}{\langle n \rangle^{2\nu-6\delta}} \right)^{\frac{1}{2}}. \end{aligned}$$

Summing this requires $\nu > \frac{1}{2}$ which restricts us further to enforcing

$$\alpha + s > \frac{11}{6},$$

completing the proof for \mathcal{N}_1 .

Now, recall from the proof of Proposition 4.1 that we obtained the estimate

$$|\mathcal{R}_1(w)| \lesssim \|\mathbf{P}_{\leq N} w\|_{H^{s-\frac{1}{2}-\varepsilon}}^6, \quad (5.7)$$

for $s > \max\left(\frac{5}{4} - \frac{1}{2}\alpha, \frac{2}{3}\right)$ and $\alpha > \frac{1}{2}$. Since

$$\max\left(\frac{2}{3}, \frac{11}{6} - \alpha, \frac{5}{4} - \frac{1}{2}\alpha\right) = \max\left(\frac{2}{3}, \frac{11}{6} - \alpha\right)$$

for any $\alpha \in \mathbb{R}$, then we may simply use (1.19) on two factors of (5.7) to obtain (5.4) for \mathcal{R}_1 . This completes the proof of (5.4). \square

5.2. Construction of weighted Gaussian measures

In this section, we construct weighted Gaussian measures which are adapted to the modified energy $E_s(v)$. Our attention is only on the low regularity setting $\frac{1}{2} < s \leq 1$ and high enough dispersion ($\alpha \geq \frac{5}{6}$), since the results in Section 4 established quasi-invariance when $s > 1$ for any $\alpha > \frac{1}{2}$.

Given $r > 0$ and $N \geq 1$, we first wish to construct the measure

$$d\rho_{s,N,r}(v) = Z_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v)$$

and then, by taking $N \rightarrow \infty$, construct the measure

$$d\rho_{s,r}(v) = Z_{s,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_s(v)} d\mu_s(v),$$

where we recall

$$R_s(v) = -\frac{1}{2} \operatorname{Re} \sum_{\Gamma(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} v_{n_1} \bar{v}_{n_2} v_{n_3} \bar{v}_{n_4},$$

and we define

$$R_{s,N}(v) := -\frac{1}{2} \operatorname{Re} \sum_{\Gamma_N(\bar{n})} \frac{\psi_s(\bar{n})}{\phi(\bar{n})} v_{n_1} \bar{v}_{n_2} v_{n_3} \bar{v}_{n_4}.$$

We set

$$F_{N,r}(v) = \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_{s,N}(\mathbf{P}_{\leq N} v)} \quad \text{and} \quad F_r(v) = \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2} R_s(v)}.$$

The main result of this subsection is the following proposition which states, not only that the probability measure $\rho_{s,r}$ exists, but that we have ‘good’ uniform L^p bounds on the density for $\rho_{s,N,r}$ (see (5.8) below). Such higher L^p bounds are crucial for the hybrid argument in [19] (see Lemma 5.5 and Proposition 5.7).

Proposition 5.2. *Let $r > 0$, $\alpha \geq \frac{3}{4}$ and $\max(\frac{5-4\alpha}{2}, \frac{1}{2}) < s \leq 1$. Then, given $p < \infty$, there exists $C > 0$ such that*

$$\|F_r(v)\|_{L^p(\mu_s)}, \|F_{N,r}(v)\|_{L^p(\mu_s)} \leq C_{p,r,s,\alpha}, \quad (5.8)$$

uniformly in $N \in \mathbb{N}$. Furthermore, there exists $R_s(v) \in L^p(\mu_s)$ such that

$$\lim_{N \rightarrow \infty} R_{s,N}(\mathbf{P}_{\leq N} v) = R_s(v) \quad \text{in } L^p(\mu_s) \quad (5.9)$$

and

$$\lim_{N \rightarrow \infty} F_{N,r}(v) = F_r(v) \quad \text{in } L^p(\mu_s). \quad (5.10)$$

In order to prove Proposition 5.2 by employing the argument in [33, Proposition 6.2], we need the following bound. Note that we define $R_{s,\infty}(v) = R_s(v)$.

Lemma 5.3. *Let $\alpha > \frac{1}{2}$ and $\frac{1}{2} < s \leq 1$. Then for any*

$$\gamma > \max\left(0, \frac{2s+1-2\alpha}{3}, \frac{1}{4} + s - \alpha\right),$$

we have

$$|R_{s,N}(\mathbf{P}_{\leq N} v)| \lesssim \|\mathbf{P}_{\leq N} v\|_{L^2} \|\mathbf{P}_{\leq N} v\|_{H^\gamma}^3, \quad (5.11)$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$. In particular, if $\alpha > \frac{3}{2}$, we may take $\gamma \equiv 0$ in (5.11).

Proof. Notice that we have a symmetry with respect to the interchange of n_1 and n_3 and the interchange of n_2 and n . We split the proof of (5.11) into a few cases with the remaining cases following analogously by exploiting this symmetry. Below we prove (5.11) for $N = \infty$ as it is clear how to adjust the argument when $N \in \mathbb{N}$. We write $\tilde{v}_n := \langle n \rangle^\gamma v_n$.

• **Case 1:** $|n_1| \sim |n| \gg |n_2|, |n_3|$

In this case, it is clear from Lemma 2.3 that

$$|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha-1} |n - n_1|,$$

and from the mean value theorem,

$$|\psi(\bar{n})| \lesssim n_{\max}^{2s-1} |n - n_1|.$$

Hence by Cauchy-Schwarz,

$$|R_s(v)| \lesssim \left(\sum_{\Gamma(\bar{n})} \frac{\tilde{v}_{n_1}^2}{n_{\max}^{2(2\alpha-2s+2\gamma)} \langle n_3 \rangle^{2\gamma}} \right)^{\frac{1}{2}} \|v\|_{H^\gamma}^2 \|v\|_{L^2} \lesssim \|v\|_{L^2} \|v\|_{H^\gamma}^3,$$

provided

$$2(2\alpha - 2s + 2\gamma) > 1 \quad \text{and} \quad 2\alpha - 2s + 3\gamma > 1. \quad (5.12)$$

• **Case 2:** $|n_2| \sim |n| \gg |n_1|, |n_3|$

Using $|\phi(\bar{n})| \gtrsim n_{\max}^{2\alpha}$ and applying Cauchy-Schwarz as in the previous case, we obtain (5.11) provided γ satisfies (5.12).

• **Case 3:** $|n_1| \sim |n_2| \sim |n_3| \sim |n|$

On $\Gamma(\bar{n})$, we have

$$|\psi_s(\bar{n})| \lesssim |n - n_3| n_{\max}^{2s-1} \quad \text{and} \quad |\phi(\bar{n})| \gtrsim |n - n_3| |n - n_1| n_{\max}^{2\alpha-2}.$$

With $\gamma \geq 0$ to be determined, we have

$$\begin{aligned} |R_s(v)| &\lesssim \sum_{\Gamma(\bar{n})} \frac{n_{\max}^{2s-1}}{|n - n_1| n_{\max}^{2\alpha-2}} \frac{\tilde{v}_{n_1} v_{n_2} \tilde{v}_{n_3} \tilde{v}_n}{\langle n_1 \rangle^\gamma \langle n_3 \rangle^\gamma \langle n \rangle^\gamma} \\ &\lesssim \sum_{\Gamma(\bar{n})} \frac{1}{|n - n_1| n_{\max}^\gamma} \tilde{v}_{n_1} v_{n_2} v_{n_3} \tilde{v}_n, \end{aligned}$$

where $v = 2\alpha - 1 - 2s + 3\gamma > 0$ provided

$$\gamma > \max\left(0, \frac{2s + 1 - 2\alpha}{3}\right).$$

By the Cauchy-Schwarz inequality and Lemma 2.4, we have

$$\begin{aligned} & \lesssim \left(\sum_{n, n_1, n_3} \frac{\tilde{v}_n^2 \tilde{v}_{n_3}^2}{\langle n - n_1 \rangle^{1+\delta}} \right)^{\frac{1}{2}} \left(\sum_{n_2, n_1, n} \frac{v_{n_2}^2 \tilde{v}_{n_1}^2}{\langle n - n_3 \rangle^{1-\delta} \langle n \rangle^{2\delta}} \right)^{\frac{1}{2}} \\ & \lesssim \|v\|_{L^2} \|v\|_{H^\gamma}^3. \end{aligned}$$

Notice from the condition $2\alpha - 1 - 2s + 2\gamma > 0$, that if $\alpha > \frac{3}{2}$, we can take $\gamma = 0$. This completes the proof. \square

We also require the following probabilistic estimate, see [33, Lemma 6.4].

Lemma 5.4. *Let $\{g_n\}_{n \in \mathbb{Z}}$ be independent standard complex-valued Gaussian random variables. Then, there exist $c, C > 0$ such that, for any $M \geq 1$, we have*

$$\mathbb{P}\left[\left(\sum_{n=1}^M |g_n|^2\right)^{\frac{1}{2}} \geq K\right] \leq e^{-cK^2},$$

provided $K \geq CM^{\frac{1}{2}}$.

We now give the proof of Proposition 5.2.

Proof of Proposition 5.2. For $\alpha > \frac{3}{2}$, Lemma 5.3 implies we may take $\gamma \equiv 0$ in (5.11) and hence

$$\mathbf{1}_{\{\|v\|_{L^2} \leq r\}} |R_{s,N}(\mathbf{P}_{\leq N} v)| \lesssim \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \|\mathbf{P}_{\leq N} v\|_{L^2}^4 \lesssim r^4,$$

at which point, the bound (5.8) follows trivially. We make up the remaining case $\frac{3}{4} \leq \alpha \leq \frac{3}{2}$ in the following. Given $\frac{1}{2} < s \leq 1$, let γ be as in Lemma 5.3 whose precise value will be specified later. On $\{\|v\|_{L^2} \leq r\}$, (5.11) implies

$$|R_s(\mathbf{P}_{\leq M_0} v)| \leq C_0 r \|\mathbf{P}_{\leq M_0} v\|_{H^\gamma}^3 \leq C_0 M_0^{3\gamma} r^4.$$

We have

$$\|F_r(v)\|_{L^p(d\mu_s)}^p \leq C^p + p \int_{\max(e, e^{2^{\frac{3}{2}} C_0 r})}^{\infty} \lambda^{p-1} \mu_s(|R_s(v)| \geq \log \lambda, \|v\|_{L^2} < r) d\lambda. \quad (5.13)$$

We choose $M_0 > 0$ such that

$$\log \lambda = 2^{\frac{3}{2}} C_0 M_0^{3\gamma} r^4. \quad (5.14)$$

For $j \in \mathbb{N}$, let $M_j = 2^j M_0$ and $\sigma_j = C_\varepsilon 2^{-\varepsilon j} = C M_0^\varepsilon M_j^{-\varepsilon}$ for some small $\varepsilon > 0$ such that $\sum_{j=1}^{\infty} \sigma_j = \frac{1}{2}$. Then we have

$$\begin{aligned} \mu_s(|R_s(v)| \geq \log \lambda, \|v\|_{L^2} < r) & \leq \mu_s(\|v\|_{H^\gamma}^2 \geq (C_0^{-1} r^{-1} \log \lambda)^{\frac{2}{3}}) \\ & \leq \sum_{j=1}^{\infty} \mu_s(\|\mathbf{P}_{M_j} v\|_{H^\gamma}^2 \geq \sigma_j (C_0^{-1} r^{-1} \log \lambda)^{\frac{2}{3}}) \\ & \lesssim \sum_{j=1}^{\infty} \mathbb{P}\left(\left(\sum_{|n| \sim M_j} |g_n|^2\right)^{\frac{1}{2}} \gtrsim L_j\right), \end{aligned}$$

where $L_j := (C_0^{-1} r^{-1} \log \lambda)^{\frac{1}{3}} \sigma_j^{\frac{1}{2}} M_j^{s-\gamma} \gtrsim M_0^{\frac{1}{3}} M_j^{s-\gamma-\frac{1}{2}\varepsilon} \gg M_j^{\frac{1}{2}}$, provided $s - \gamma > \frac{1}{2}$. We used here that $\lambda > e^{2^{\frac{3}{2}} C_0 r}$ implies, from (5.14), $M_0^\gamma r \geq 1$ and hence $(C_0^{-1} r^{-1} \log \lambda)^{2/3} \sim M_0^{4\gamma} r^4 \gtrsim 1$. Therefore, by Lemma 5.4, we have

$$\begin{aligned} \mu_s(|R_s(v)| \geq \log \lambda, \|v\|_{L^2} < r) &\lesssim \sum_{j=1}^{\infty} e^{-c_r 2^{j(2s-2\gamma-\frac{2}{3}\varepsilon)} (\log \lambda)^{\frac{2}{3} + \frac{2}{3} \frac{s-\gamma}{\gamma}}} \\ &\lesssim e^{-c_r'' (\log \lambda)^{\frac{2s}{3\gamma}}}. \end{aligned}$$

Thus, from (5.13), we have

$$\|F_r(v)\|_{L^p(d\mu_s)}^p \lesssim C_p + p \int_C^{\infty} e^{p\lambda} e^{-c_r'' \lambda^{\frac{2s}{3\gamma}}} d\lambda < C < \infty,$$

provided $\frac{2}{3}s > \gamma$. It is clear that the above arguments also apply to obtain the uniform bound (5.8) when $N \in \mathbb{N}$.

Thus we can construct the measure $\rho_{s,r}$ provided we can choose $\gamma \in \mathbb{R}$ satisfying

$$\max\left(0, \frac{2s+1-2\alpha}{3}, \frac{1}{4} + s - \alpha\right) < \gamma < \min\left(s - \frac{1}{2}, \frac{2}{3}s\right) = s - \frac{1}{2}.$$

As we wish to consider s close to 1, we must impose $\alpha > \frac{3}{4}$ to rule out the maximum on the left hand side being $\frac{1}{4} + s - \alpha$. Now, if $s + \frac{1}{2} - \alpha \leq 0$, it is clear we can pick a $\gamma > 0$. Otherwise, if $s + \frac{1}{2} - \alpha \geq 0$, we can choose $\gamma > 0$ as long as

$$\frac{2s+1-2\alpha}{3} < s - \frac{1}{2},$$

which upon rearranging yields the condition $\max(\frac{5-4\alpha}{2}, \frac{1}{2}) < s \leq 1$.

As for the $L^p(\mu_s)$ convergence of $R_{s,N}(\mathbf{P}_{\leq N}v)$ and $F_{N,r}(v)$, we note that when $\alpha > \frac{3}{2}$, we have

$$|R_{s,N}(\mathbf{P}_{\leq N}(v)) - R_s(v)| \lesssim \|\mathbf{P}_{>N}v\|_{L^2} \|v\|_{L^2}^3. \quad (5.15)$$

By a slight modification of (5.11), when $\frac{1}{2} < \alpha < \frac{3}{2}$, we also have

$$|R_{s,N}(\mathbf{P}_{\leq N}(v)) - R_s(v)| \lesssim \|\mathbf{P}_{>N}v\|_{L^2} \|v\|_{H^\gamma}^3 + \|v\|_{L^2} \|v\|_{H^\gamma}^2 \|\mathbf{P}_{>N}v\|_{H^\gamma}. \quad (5.16)$$

Taking $N \rightarrow \infty$ in (5.15) and (5.16) and noting that $s - \gamma > \frac{1}{2}$ shows $R_{s,N}(\mathbf{P}_{\leq N}v)$ converges almost surely with respect to μ_s to $R_s(v)$. Then because of the uniform in N bounds

$$\|R_{s,N}(\mathbf{P}_{\leq N}v)\|_{L^p(\mu_s)}, \|R_s(v)\|_{L^p(\mu_s)} \leq C_{p,s} < \infty,$$

which follow from Lemma 5.3, a standard argument using Egoroff's theorem implies (5.9) (see [33, Proposition 6.2]).

From (5.15) and (5.16), we have almost sure convergence of $F_{r,N}(v)$ to $F_r(v)$ with respect to μ_s . Using (5.8), the above standard argument implies convergence in $L^p(\mu_s)$; namely (5.10). This completes the proof of Proposition 5.2. \square

The next lemma shows that the two factors lying in $\mathcal{FL}^{\sigma,\infty}$ for $\sigma < s$ in the modified energy estimate (5.4) have moments indeed contributing a factor of $p^{\frac{1}{2}}$. Notice that as a consequence of Proposition 5.2 (namely, $Z_{s,N,r} \rightarrow Z_{s,r}$ as $N \rightarrow \infty$), $Z_{s,N,r}^{-1}$ is bounded uniformly in $N \in \mathbb{N}$.

Lemma 5.5. *Given $\varepsilon > 0$ and $r > 0$, there exists $C = C(\varepsilon, r) > 0$ such that*

$$\| \|f\|_{\mathcal{FL}^{s-\varepsilon,\infty}} \|_{L^p(\rho_{s,N,r})} \leq Cp^{\frac{1}{2}}$$

for any $p \geq 1$ and $N \in \mathbb{N}$.

Proof. Applying the uniform bound (5.8), the uniform bound on $Z_{s,N,r}^{-1}$ and Minkowski's integral inequality, for any $q > \frac{1}{\varepsilon}$, we have

$$\begin{aligned} \left\| \|f\|_{\mathcal{F}L^{s-\varepsilon,\infty}} \right\|_{L^p(\rho_{s,N,r})} &\leq \left\| \|f\|_{\mathcal{F}L^{s-\varepsilon,q}} \right\|_{L^p(\rho_{s,N,r})} \\ &\leq Z_{s,N,r}^{-\frac{1}{p}} \|F_{N,r}\|_{L^{q'}(\mu_s)}^{\frac{1}{p}} \left\| \|f\|_{\mathcal{F}L^{s-\varepsilon,q}} \right\|_{L^{pq}(\mu_s)}^{\frac{1}{p}} \\ &\lesssim_q \left\| \langle n \rangle^{-\varepsilon} \|g_n\|_{L^{pq}(\Omega)} \right\|_{\ell_n^q} \\ &\lesssim_q q^{\frac{1}{2}} p^{\frac{1}{2}}, \end{aligned}$$

where we have used the following well-known estimate on higher moments of Gaussian random variables in the last inequality:

$$\|g_n\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} \quad (5.17)$$

for any $p \geq 2$. \square

5.3. Transport of the truncated weighted Gaussian measures

In this subsection, we study how the measures $\rho_{s,N,r}$ evolve under the flow of the truncated equation (3.2). We follow the method of [19] in which we use a ‘change of variables formula’ (see Lemma 5.6) to make the modified energy E_s along the truncated flow appear. Taking a time derivative and using the estimate (5.4) then gives a differential inequality for the evolution of $\rho_{s,N,r}$ under Φ_N (see Proposition 5.7).

Lemma 5.6 (Change of variables formula). *Let $\alpha \geq \frac{5}{6}$, s be as in (5.1) and $r > 0$. Then for any $N \in \mathbb{N}$, $t \in \mathbb{R}$ and measurable set $A \subset H^{s-\frac{1}{2}-\varepsilon}(\mathbb{T})$, we have*

$$\rho_{s,N,r}(\Phi_N(t)(A)) = \widehat{Z}_{s,N}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v))} du_{\leq N} \times d\mu_{s,N}^{\perp}. \quad (5.18)$$

We omit the proof of Lemma 5.6 as it is identical to those in [19,33,35]. The core ingredients are the invariance of the Lebesgue measure L_N under the truncated flow Φ_N (because of Liouville's theorem), the invariance of Φ_N in the L^2 -norm (mass conservation) and the bijectivity of the flow Φ_N . When $\alpha \geq 1$, (5.18) also holds for any measurable $A \subset L^2(\mathbb{T})$.

Proposition 5.7. *Let $\alpha \geq \frac{5}{6}$ and s be as in (5.1). Then, given $r, R > 0$ and $T > 0$, there exists $C_{r,R,T} > 0$ such that*

$$\frac{d}{dt} \rho_{s,N,r}(\Phi_N(t)(A)) \leq C_{r,R,T} \cdot p \{\rho_{s,N,r}(\Phi_N(t)(A))\}^{1-\frac{1}{p}} \quad (5.19)$$

for any $p \geq 2$, any $N \in \mathbb{N}$, any $t \in [0, T]$ and any measurable set $A \subset B_R \subset H^{\sigma}(\mathbb{T})$.

Proof. Fix $r, R > 0$ and $T > 0$. As a preliminary step, we first note the following key estimate on the growth of the modified energy $E_{s,N}$: for α and s as in Proposition 5.7, we have

$$\left\| \mathbf{1}_{B_R} \partial_t E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v)) \right\|_{t=0} \left\|_{L^p(\rho_{s,N,r})} \leq C_{r,R} \cdot p \quad (5.20)$$

for any $p \geq 2$ and for any $N \in \mathbb{N}$. This follows from (5.4), Lemma 5.5, the uniform bound (5.8) on $F_{N,r}$, the uniform bound on $Z_{s,N,r}^{-1}$ and Cauchy-Schwarz inequality, since

$$\begin{aligned} \text{LHS of (5.20)} &\leq Z_{s,N,r}^{-\frac{1}{p}} \|F_{N,r}(v)\|_{L^2(\mu_s)}^{\frac{1}{p}} \left\| \mathbf{1}_{B_R} \partial_t E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v)) \right\|_{t=0} \|_{L^{2p}(\mu_s)} \\ &\leq C \left\| \mathbf{1}_{B_R} \right\|_{\mathcal{F}L^{s-\frac{\varepsilon}{2},\infty}}^2 \left\| \mathbf{P}_{\leq N} v \right\|_{H^{\sigma}}^4 \|_{L^{2p}(\mu_s)} \end{aligned}$$

$$\begin{aligned} &\leq CR^4 \left\| \mathbf{P}_{\leq N} v \right\|_{\mathcal{FL}^{s-\frac{\varepsilon}{2}, \infty}}^2 \left\| \right\|_{L^4 p(\mu_s)}^2 \\ &\leq CR^4 p. \end{aligned}$$

Now fix a measurable set $A \subset B_R \subset H^\sigma(\mathbb{T})$ and $t_0 \in [0, T]$. By the semigroup property of $\Phi_N(t)$ and the change of variables formula (Lemma 5.6), we have

$$\begin{aligned} \left. \frac{d}{dt} \rho_{s,N,r}(\Phi_N(t)(A)) \right|_{t=t_0} &= Z_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \Big|_{t=t_0} \\ &= Z_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t)(\Phi_N(t_0)(A))} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \Big|_{t=0} \\ &= \widehat{Z}_{s,N,r}^{-1} \frac{d}{dt} \int_{\Phi_N(t_0)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v))} du_{\leq N} \times d\mu_{s,N}^\perp \Big|_{t=0} \\ &= -Z_{s,N,r}^{-1} \int_{\Phi_N(t_0)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} \partial_t E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v))|_{t=0} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v). \end{aligned}$$

Now recall from Proposition 4.5 (i) that for any $t \in [0, T]$ and $N \in \mathbb{N}$, there exists $C(R, T) > 0$ such that $\Phi_N(t)(B_R) \subset B_{C(R, T)}$. Note that when the flow Φ is only well-defined locally-in-time, we use Proposition 4.5 (i). Hence, by Hölder's inequality we obtain

$$\begin{aligned} \left. \frac{d}{dt} \rho_{s,N,r}(\Phi_N(t)(A)) \right|_{t=t_0} &\leq Z_{s,N,r}^{-1} \int_{\Phi_N(t_0)(A)} |\partial_t E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(u))|_{t=0} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-R_{s,N}(\mathbf{P}_{\leq N} v)} d\mu_s(v) \\ &\leq \left\| \mathbf{1}_{B_{C(R, T)}} \partial_t E_{s,N}(\mathbf{P}_{\leq N} \Phi_N(t)(v)) \right\|_{L^p(\rho_{s,N,r})} \Big|_{t=0} \left\| \rho_{s,N,r}(\Phi_N(t_0)(A)) \right\|^{1-\frac{1}{p}}. \end{aligned}$$

Applying (5.20) yields (5.19). \square

5.4. Proof of Theorem 1.2 (i)

In this section, we apply the argument in [35] to deduce from Proposition 5.7, the quasi-invariance of μ_s under the untruncated flow $\Phi(t)$. In what follows, we fix $\alpha \geq \frac{5}{6}$ and consider s satisfying (5.1). We show that for each fixed $R > 0$,

$$\text{if } \mu_s(A) = 0, \quad \text{then} \quad \mu_s(\Phi(t)(A)) = 0 \quad (5.21)$$

for any $t \in [0, T(R)]$ and for any measurable set $A \subset B_R$. This implies local-in-time quasi-invariance of μ_s under (1.1) for any

$$s > \frac{11}{6} - \alpha.$$

When $\alpha \geq 1$, (5.21) is true for any $t \in \mathbb{R}$. As R is arbitrary, this implies quasi-invariance of μ_s under the dynamics of FNLS (1.1). For the rest of this section, we fix $R > 0$ and $\alpha \geq 1$ since the arguments below are easily modified to imply local-in-time quasi-invariance when $\frac{5}{6} < \alpha < 1$.

For the first step, we use Proposition 5.7 to show that $\rho_{s,N,r}$ is quasi-invariant under $\Phi_N(t)$; see Lemma 5.8. The proof of Lemma 5.8 follows exactly as in [35, Proposition 5.3].

Lemma 5.8. *Given $r > 0$, there exists $0 < t_{r,R} < T$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that if, for a measurable set $A \subset B_R \subset H^\sigma(\mathbb{T})$, there exists $N_0 \in \mathbb{N}$ such that*

$$\rho_{s,N,r}(A) < \delta$$

for any $N \geq N_0$, then we have

$$\rho_{s,N,r}(\Phi_N(t)(A)) < \varepsilon$$

for any $t \in [0, t_{r,R}]$ and any $N \geq N_0$.

Then a careful argument allows the previous statement to hold when $N = \infty$; that is, we have that $\rho_{s,r}$ is quasi-invariant under the untruncated flow $\Phi(t)$ (Lemma 5.9). The proof of Lemma 5.9 makes use of the approximation property of the dynamics of FNLS (1.1) as in Proposition 4.5 (ii) and follows the arguments in [35, Lemma 5.5].

Lemma 5.9. *Given $r > 0$, there exists $0 < t_{r,R} < T$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\rho_{s,r}(A) < \delta,$$

then we have

$$\rho_{s,r}(\Phi_N(t)(A)) < \varepsilon$$

for any $t \in [0, t_{r,R}]$.

Now, invoking the mutual absolute continuity of $\rho_{s,r}$ and $\mu_{s,r}$ implies $\mu_{s,r}$ is quasi-invariant under $\Phi(t)$. We then take $r \rightarrow \infty$ (as in [35, Theorem 1.2]) and iterate in time to obtain (5.21) for every $t \in \mathbb{R}$, for a fixed $R > 0$. This concludes the proof of Theorem 1.2 (i).

Declaration of competing interest

There is no competing interest.

Acknowledgements

J. F. and W. J. T. would like to thank their advisors, Tadahiro Oh and Oana Pocovnicu, for suggesting this problem and their continuous guidance and support throughout. J. F. and W. J. T. were supported by The Maxwell Institute Graduate School in Analysis and its Applications, a Centre for Doctoral Training funded by the UK Engineering and Physical Sciences Research Council (grant EP/L016508/01), the Scottish Funding Council, Heriot-Watt University and the University of Edinburgh. J. F. also acknowledges support from Tadahiro Oh's ERC starting grant (no. 637995 "ProbDynDispEq").

Appendix A. Proof of Lemma 2.3 for $\alpha > 1$

Setting $k = n_1 - n$ and $j = n_3 - n$, it is equivalent to prove

$$g(j, k, n) := ||n + k|^{2\alpha} - |n + k + j|^{2\alpha} + |n + j|^{2\alpha} - |n|^{2\alpha}| \gtrsim |k||j|(|k| + |j| + |n|)^{2\alpha-2}.$$

Since $2\alpha > 2$, the function $f(x) = |x|^{2\alpha} \in C^2(\mathbb{R})$ and satisfies

$$f'(x) = 2\alpha|x|^{2\alpha-2}x, \quad f''(x) = 2\alpha(2\alpha-1)|x|^{2\alpha-2}.$$

We follow a similar argument to the case $\frac{1}{2} < \alpha < 1$ in [11]. Without loss of generality we can assume that $\max(|j|, |k|) = |j|$ and $j \neq 0$. For any $c \in \mathbb{R}$, define $f_c(x) := |x + c|^{2\alpha} - |x - c|^{2\alpha}$. Then, we have

$$g(j, k, n) = |f_{j/2}(n + j/2) - f_{j/2}(n + k + j/2)|.$$

The mean value theorem implies

$$g(j, k, n) \gtrsim |k| \min_{x \in I} |f'_{j/2}(x)|,$$

where I is either the interval $(n + j/2, n + j/2 + k)$ or the interval $(n + j/2 + k, n + j/2)$. It suffices to show

$$|f'_c(x)| \gtrsim |c| \max(|x|, |c|)^{2\alpha-2} \text{ for } |c| \geq \frac{1}{2}. \quad (\text{A.1})$$

To see this, we first suppose $|n| \lesssim |j|$. Then, for any $x \in I$, we have

$$|f'_{j/2}(x)| \gtrsim |j|^{2\alpha-1} \gtrsim |j|(|k| + |j| + |n|)^{2\alpha-2}.$$

Now suppose $|n| \gg |j|$. Then $x \in I$ implies $|x| \sim |n|$ and hence

$$\min_{x \in I} |f'_{j/2}(x)| \gtrsim |j||n|^{2\alpha-2} \gtrsim |j|(|k| + |j| + |n|)^{2\alpha-2}.$$

In order to verify (A.1), we may assume that $x \geq 0$ as f_c is odd and similarly, we assume $c \geq \frac{1}{2}$ as f'_c is odd in c . We have

$$f'_c(x) = 2\alpha|x + c|^{2\alpha-2}(x + c) - 2\alpha|x - c|^{2\alpha-2}(x - c),$$

and we consider three cases.

• **Subcase 2.1:** $0 \leq x \leq c$

Here we have

$$f'_c(x) = 2\alpha \left[|x + c|^{2\alpha-1} + |x - c|^{2\alpha-1} \right] \gtrsim c^{2\alpha-1}.$$

• **Subcase 2.2:** $c < x \leq 2c$

We have

$$\begin{aligned} f'_c(x) &= 2\alpha \left[(x + c)^{2\alpha-1} - (x - c)^{2\alpha-1} \right] \\ &= 2\alpha c^{2\alpha-1} \left[\left(\frac{x}{c} + 1 \right)^{2\alpha-1} - \left(\frac{x}{c} - 1 \right)^{2\alpha-1} \right] \gtrsim c^{2\alpha-1} \left(\frac{x}{c} \right)^{2\alpha-2} \sim cx^{2\alpha-2}. \end{aligned}$$

• **Subcase 2.3:** $x > 2c$

Using the mean value theorem, we have

$$f'_c(x) = 2\alpha x^{2\alpha-1} \left[\left(1 + \frac{c}{x} \right)^{2\alpha-1} - \left(1 - \frac{c}{x} \right)^{2\alpha-1} \right] \gtrsim x^{2\alpha-1} \frac{c}{x} \sim cx^{2\alpha-2}.$$

This completes the proof of (A.1).

Appendix B. Well-posedness theory for FNLS with $\alpha > 1$ in $L^2(\mathbb{T})$

In this appendix, we detail the global well-posedness of the Cauchy problem:

$$\begin{cases} i \partial_t u + (-\partial_x^2)^\alpha u = \pm |u|^2 u \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}), \end{cases} \quad (\text{B.1})$$

with $\alpha > 1$ and for any $s \geq 0$. We say $u \in C([0, T]; H^s(\mathbb{T}))$ is a solution to (B.1) if it satisfies the following integral (Duhamel) formulation

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t') |u(t')|^2 u(t') dt',$$

where $S(t) = e^{-it(-\partial_x^2)^\alpha}$. The main result is the following:

Proposition B.1. *Let $s \geq 0$ and $\alpha > 1$. Then, given $u_0 \in H^s(\mathbb{T})$, there exist $T = T(\|u_0\|_{L^2}) > 0$ and a unique solution $u \in C([-T, T]; H^s(\mathbb{T}))$ to (B.1) with $u|_{t=0} = u_0$. Furthermore, we have*

$$\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \leq C \|u_0\|_{H^s}.$$

While the overall arguments presented here are standard, to our knowledge, they have not been fully explicated in the literature for (B.1) apart from when $\alpha = 2$, see [33, Appendix A]. We follow the ideas presented in [11], which considered $\frac{1}{2} < \alpha < 1$. The crucial ingredient is the lower bound on the phase function $\phi(\overline{n})$ given in Lemma 2.3. As the global well-posedness for $s > 0$ will follow by iterating the local argument using the mass conservation (see (1.2)), and the local argument is independent of the defocusing or focusing nature of (B.1), we may assume (B.1) is defocusing (that is, the sign on the nonlinearity in (B.1) is positive).

We will show the well-posedness of the gauged equation

$$i\partial_t v + (-\partial_x^2)^\alpha v = \left(|v|^2 - 2 \int_{\mathbb{T}} |v|^2 dx \right) v, \quad (\text{B.2})$$

in $H^s(\mathbb{T})$ for $s \geq 0$. This suffices from the fact that given the solution $v \in C(\mathbb{R}; L^2(\mathbb{T}))$ satisfying $v|_{t=0} = u_0$ to (B.2), the function

$$u(t) = \mathcal{G}^{-1}[v](t) = e^{-2it \int |v|^2 dx} v$$

is the solution to (B.1) with $u|_{t=0} = u_0$ (see Section 3). We define the non-resonant and resonant operators (respectively) by

$$\begin{aligned} \mathcal{N}(v_1, v_2, v_3) &= \sum_{n \in \mathbb{Z}} e^{inx} \sum_{\Gamma(n)} \widehat{v}_1(n_1) \overline{\widehat{v}_2(n_2)} \widehat{v}_3(n_3) \text{ and} \\ \mathcal{R}(v_1, v_2, v_3) &= - \sum_{n \in \mathbb{Z}} e^{inx} \widehat{v}_1(n) \overline{\widehat{v}_2(n)} \widehat{v}_3(n), \end{aligned} \quad (\text{B.3})$$

where for fixed $n \in \mathbb{Z}$, the set $\Gamma(n)$ is the non-resonant hyperplane given in (3.5). We will write $\mathcal{N}(v, v, v) = \mathcal{N}(v)$ and the same for \mathcal{R} . Then, the integral formulation of (B.2) becomes

$$v(t) = S(t)u_0 - i \int_0^t S(t-t')(\mathcal{N}(v)(t') + \mathcal{R}(v)(t'))dt' =: \Lambda(v)(t). \quad (\text{B.4})$$

Our goal is to prove local well-posedness of (B.2) via a contraction mapping argument for the operator Λ in the Fourier-restriction spaces $X^{s,b}(\mathbb{R} \times \mathbb{T})$. We now state some basic properties of the spaces $X^{s,b}(\mathbb{R} \times \mathbb{T})$. Given $s, b \in \mathbb{R}$, we define the space $X^{s,b}(\mathbb{R} \times \mathbb{T})$ via the norm

$$\|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{T})} = \|\langle n \rangle^s \langle \tau - |n|^{2\alpha} \rangle^b \widehat{v}(\tau, n)\|_{L_t^2 \ell_n^2(\mathbb{R} \times \mathbb{Z})},$$

where $\widehat{v}(\tau, n)$ denotes the space-time Fourier transform of $v(t, x)$. Given $T > 0$, we also define the local-in-time version $X^{s,b}([0, T] \times \mathbb{T})$ of $X^{s,b}(\mathbb{R} \times \mathbb{T})$ as

$$X^{s,b}([0, T] \times \mathbb{T}) = \inf\{\|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{T})} : v|_{[0,T]} = u\}.$$

We will denote by $X^{s,b}$ and $X_T^{s,b}$ the spaces $X^{s,b}(\mathbb{R} \times \mathbb{T})$ and $X^{s,b}([0, T] \times \mathbb{T})$, respectively. We have the following embedding: for any $s \in \mathbb{R}$ and $b > \frac{1}{2}$, we have

$$X_T^{s,b} \hookrightarrow C([0, T]; H^s(\mathbb{T})). \quad (\text{B.5})$$

Given any function F on $[0, T] \times \mathbb{T}$, we denote by \tilde{F} any extension of F onto $\mathbb{R} \times \mathbb{T}$.

The following linear estimates hold in $X_T^{s,b}$. Their proofs are standard and can be found in, for example, [18].

Lemma B.2. *The following are true:*

(i) [Homogeneous linear estimate] *Given $s, b \in \mathbb{R}$, we have*

$$\|S(t)v\|_{X_T^{s,b}} \lesssim \|v\|_{H^s}, \quad (\text{B.6})$$

for any $0 < T \leq 1$.

(ii) [Nonhomogeneous linear estimate] Let $s \in \mathbb{R}$ and $-\frac{1}{2} < b' \leq 0 \leq b \leq 1 + b'$. Then we have

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{X_T^{s,b}} \lesssim T^{1+b'-b} \|F\|_{X_T^{s,b'}}, \quad (\text{B.7})$$

for any $0 < T \leq 1$.

We now state the crucial nonlinear estimate for the operators of (B.3).

Proposition B.3. Let $\alpha > 1$ and $s \geq 0$. Then for $b > \frac{1}{2}$, there exists $b' < \frac{1}{2}$, sufficiently close to $\frac{1}{2}$, such that we have

$$\|\mathcal{R}(v_1, v_2, v_3)\|_{X_T^{s,-b'}} + \|\mathcal{N}(v_1, v_2, v_3)\|_{X_T^{s,-b'}} \lesssim \min_{k \in \{1,2,3\}} \left(\|v_k\|_{X_T^{s,b}} \prod_{\substack{j=1 \\ j \neq k}}^3 \|v_j\|_{X_T^{0,b}} \right) \quad (\text{B.8})$$

for any $0 < T \leq 1$.

Proof. We follow the proof in [11, Proposition 5], but since $\alpha > 1$, the proof here is simpler. Let \tilde{v}_j be extensions of v_j , for $j = 1, 2, 3$. Then it suffices to prove

$$\|\mathcal{N}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)\|_{X^{s,-b'}} + \|\mathcal{R}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)\|_{X^{s,-b'}} \lesssim \min \left(\|\tilde{v}_k\|_{X^{s,b}} \prod_{\substack{j=1 \\ j \neq k}}^3 \|\tilde{v}_j\|_{X^{0,b}} \right), \quad (\text{B.9})$$

since (B.8) follows from (B.9) by taking an infimum over all extensions \tilde{v}_j of v_j . For simplicity, we write \tilde{v}_j as v_j . We begin with the estimate for \mathcal{N} . By Plancherel, we have

$$\|\mathcal{N}(v_1, v_2, v_3)\|_{X^{s,-b'}} = \left\| \int_{\tau=\tau_1-\tau_2+\tau_3} \sum_{\Gamma(n)} \frac{\langle n \rangle^s \widehat{v}_1(\tau_1, n_1) \overline{\widehat{v}_2(\tau_2, n_2)} \widehat{v}_3(\tau_3, n_3)}{\langle \tau - |n|^{2\alpha} \rangle^{b'}} d\tau_1 d\tau_2 \right\|_{L_\tau^2 \ell_n^2}.$$

Notice $|n| \lesssim \max_{j=1,2,3} |n_j|$ and thus we may suppose, without loss of generality, $|n| \lesssim |n_1|$. We define

$$f_1(\tau, n) = \langle n \rangle^s \langle \tau - |n|^{2\alpha} \rangle^b |\widehat{v}_1(\tau, n)| \quad \text{and} \quad g_j(\tau, n) = \langle \tau - |n|^{2\alpha} \rangle^b |\widehat{v}_j(\tau, n)| \quad \text{for } j = 2, 3.$$

Then we see that (B.9) follows if we prove

$$\begin{aligned} & \left\| \int_{\tau=\tau_1-\tau_2+\tau_3} \sum_{\Gamma(n)} \frac{f_1(\tau_1, n_1) g_2(\tau_2, n_2) g_3(\tau_3, n_3)}{\langle \tau - |n|^{2\alpha} \rangle^{b'}} \prod_{j=1}^3 \frac{1}{\langle \tau_j - |n_j|^{2\alpha} \rangle^b} d\tau_1 d\tau_2 \right\|_{L_\tau^2 \ell_n^2} \\ & \lesssim \|g_2\|_{L_\tau^2 \ell_n^2} \|g_3\|_{L_\tau^2 \ell_n^2} \|f_1\|_{L_\tau^2 \ell_n^2}. \end{aligned} \quad (\text{B.10})$$

By duality, we have

$$\begin{aligned} \text{LHS of (B.10)} & \lesssim \sup_{\|h\|_{L_\tau^2 \ell_n^2}=1} \int_{\tau=\tau_1+\tau_2-\tau_3=0} \sum_{\substack{n=n_1+n_2-n_3=0 \\ n_1, n_3 \neq n}} \frac{f_1(\tau_1, n_1) g_2(\tau_2, n_2) g_3(\tau_3, n_3) h(\tau, n)}{\langle \tau - |n|^{2\alpha} \rangle^{b'}} \\ & \quad \times \prod_{j=1}^3 \frac{1}{\langle \tau_j - |n_j|^{2\alpha} \rangle^b} d\tau_1 d\tau_2 d\tau. \end{aligned}$$

Applying Cauchy-Schwarz, we bound this by

$$\|h\|_{L_\tau^2 \ell_n^2} \|g_2\|_{L_\tau^2 \ell_n^2} \|g_3\|_{L_\tau^2 \ell_n^2} \|f_1\|_{L_\tau^2 \ell_n^2} \sup_{n, \tau} M_{n, \tau}^{\frac{1}{2}},$$

where

$$M_{n,\tau} = \sum_{\substack{n_1-n_2+n_3=n \\ n_1, n_3 \neq n}} \int_{\tau=\tau_1-\tau_2+\tau_3} \frac{1}{\langle \tau - |n|^{2\alpha} \rangle^{2b'}} \prod_{j=1}^3 \frac{1}{\langle \tau_j - |n_j|^{2\alpha} \rangle^{2b}} d\tau_1 d\tau_2.$$

It is then clear that we have proved the thesis if we show

$$\sup_{n,\tau} M_{n,\tau} < \infty.$$

Fix $n \in \mathbb{Z}$ and $\tau \in \mathbb{R}$. As $2b > 1$, integrating in τ_2 and τ_1 (by using Lemma 2.4 twice) gives

$$M_{n,\tau} \lesssim \sum_{\substack{n_1-n_2+n_3=n \\ n_1, n_3 \neq n}} \frac{1}{\langle \tau - |n|^{2\alpha} \rangle^{2b'}} \frac{1}{\langle \tau - |n|^{2\alpha} - \phi(\bar{n}) \rangle^{2b}} \lesssim \sum_{\substack{n_1-n_2+n_3=n \\ n_1, n_3 \neq n}} \frac{1}{\langle \phi(\bar{n}) \rangle^{2b'}}.$$

The last inequality above follows from the triangle inequality. Recalling that $n_1, n_3 \neq n$, Lemma 2.3 implies $|\phi(\bar{n})| \gtrsim 1$ and hence by Lemma 2.4, we have

$$\begin{aligned} M_{n,\tau} &\lesssim \sum_{\substack{n_1-n_2+n_3=n \\ n_1, n_3 \neq n}} \frac{1}{\langle n - n_1 \rangle^{2b'} \langle n - n_3 \rangle^{2b'} n_{\max}^{4b'(\alpha-1)}} \\ &\lesssim \left(\sum_{n_1} \frac{1}{\langle n - n_1 \rangle^{2b'} \langle n_1 \rangle^{2b'(\alpha-1)}} \right)^2 < C < \infty, \end{aligned}$$

independently of n provided we choose $b' < \frac{1}{2}$ such that

$$2b'\alpha > 1.$$

The above condition is satisfied since $\alpha > 1$. This completes the proof of (B.9) for the non-resonant operator \mathcal{N} . The case for the resonant operator \mathcal{R} is simpler. Indeed, by Young's inequality and Hölder's inequality, we have

$$\begin{aligned} \|\mathcal{R}(v_1, v_2, v_3)\|_{X^{s,-b'}} &\leq \|\mathcal{R}(v_1, v_2, v_3)\|_{X^{s,0}} = \|\langle n \rangle^s (\widehat{v}_1 *_{\tau} \overline{\widehat{v}_2} *_{\tau} \widehat{v}_3)(\tau, n)\|_{L_{\tau}^2 \ell_n^2} \\ &\lesssim \|\langle n \rangle^s \widehat{v}_1(\tau, n)\|_{L_{\tau}^2} \prod_{j=2}^3 \|\widehat{v}_j(\tau, n)\|_{L_{\tau}^1} \|\ell_n^2\|_{\ell_n^2} \\ &\lesssim \|\langle n \rangle^s \widehat{v}_1(\tau, n)\|_{L_{\tau}^2} \|\ell_n^2\|_{\ell_n^2} \prod_{j=2}^3 \|\langle \tau - |n|^{2\alpha} \rangle^b \widehat{v}_j(\tau, n)\|_{L_{\tau}^2} \|\ell_n^2\|_{\ell_n^2} \\ &\sim \|v_1\|_{X^{s,0}} \prod_{j=2}^3 \|v_j\|_{X^{0,b}} \lesssim \|v_1\|_{X^{s,b}} \prod_{j=2}^3 \|v_j\|_{X^{0,b}}. \end{aligned}$$

It is clear that the right hand side of (B.8) follows analogously. This completes the proof. \square

We now prove Proposition B.1.

Proof of Proposition B.1. Fix $\alpha > 1$ and let $u_0 \in L^2(\mathbb{T})$. For $0 < T \leq 1$, from (B.4), we have

$$\Lambda(v)(t) = \Lambda(v)_{u_0}(t) := S(t)u_0 - i \int_0^t S(t-t')(\mathcal{N}(v)(t') + \mathcal{R}(v)(t')) dt'.$$

Now let b' be given by Proposition B.3 and write $b' = \frac{1}{2-\delta}$ for some small $\delta > 0$ and then set $b = \frac{1}{2} + \frac{\delta}{2}$. Then by (B.6), (B.7) and (B.8), we have

$$\|\Lambda(v)\|_{X_T^{0,b}} \lesssim \|u_0\|_{L^2} + T^{\frac{\delta}{2}} \|v\|_{X_T^{0,b}}^3. \quad (\text{B.11})$$

Similarly,

$$\|\Lambda(v_1) - \Lambda(v_2)\|_{X_T^{0,b}} \lesssim T^{\frac{\delta}{2}} (\|v_1\|_{X_T^{0,b}}^2 + \|v_2\|_{X_T^{0,b}}^2) \|v_1 - v_2\|_{X_T^{0,b}}. \quad (\text{B.12})$$

With $R \sim 2\|u_0\|_{L^2}$, we let B_R be the closed ball of radius R in $X_T^{0,b}$. It follows by the contraction mapping theorem using (B.11) and (B.12) and choosing $T = T(\|u_0\|_{L^2}) > 0$ that Λ is a contraction over B_R and hence we have a unique fixed point $v \in X_T^{0,b}$.

Now for $s > 0$, let $u_0 \in H^s(\mathbb{T})$. Then by (B.6), (B.7) and (B.8), we obtain

$$\begin{aligned} \|\Lambda(v)\|_{X_T^{s,b}} &\lesssim \|u_0\|_{H^s} + T^{\frac{\delta}{2}} \|v\|_{X_T^{0,b}}^2 \|v\|_{X_T^{s,b}} \\ &\lesssim \|u_0\|_{H^s} + T^{\frac{\delta}{2}} \|u_0\|_{L^2}^2 \|v\|_{X_T^{s,b}}, \end{aligned} \quad (\text{B.13})$$

and

$$\|\Lambda(v_1) - \Lambda(v_2)\|_{X_T^{s,b}} \lesssim T^{\frac{\delta}{2}} \|u_0\|_{L^2}^2 \|v_1 - v_2\|_{X_T^{s,b}}. \quad (\text{B.14})$$

It is clear from (B.13) and (B.14) that we then obtain a unique solution $v \in X_T^{s,b}$ for the same $T = T(\|u_0\|_{L^2}) > 0$. Continuity of v in time follows from (B.5). This completes the local well-posedness proof. \square

We now move onto the proof of the global well-posedness of (1.1) for any $\alpha > 1$ as stated in Proposition 1.1 (i).

Proof of Proposition 1.1 (i) for $\alpha > 1$. When $s \geq \alpha$, we have for smooth solutions to (B.1) the growth bound

$$\|u(t)\|_{H^s} \leq e^{Ct} \|u_0\|_{H^s},$$

with $C = C(\|u_0\|_{H^1})$. This follows from the coercivity of the energy,⁵ the inequality

$$\| |u|^2 u \|_{H^s} \lesssim \|u\|_{H^\alpha}^2 \|u\|_{H^s}, \quad (s \geq \alpha)$$

and Gronwall's inequality.

For $s = 0$, we have the mass conservation $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$ for any $t \in \mathbb{R}$, immediately yielding global-in-time existence. For $0 < s < \alpha$, we iterate the mass conservation. The L^2 local theory (Proposition B.1) implies there is a T_0 depending on $\|u_0\|_{L^2}$ such that

$$\|v\|_{X_{T_0}^{0,b}} \lesssim \|u_0\|_{L^2}, \quad (\text{B.15})$$

for, say, $b = \frac{1}{2} +$. We have from (B.13) and (B.15), that for any $T_1 \leq T_0$,

$$\|v\|_{X_{T_1}^{s,b}} \lesssim \|u_0\|_{H^s} + T_1^\theta \|u_0\|_{L^2}^2 \|v\|_{X_{T_1}^{s,b}}.$$

Therefore, there is a $T_2(\|u_0\|_{L^2}) \leq T$ such that

$$\|v\|_{X_{T_2}^{s,b}} \lesssim \|u_0\|_{H^s}.$$

From (B.5), we get the a priori bound

$$\sup_{t \in [0, T_2]} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s},$$

which can be iterated to yield

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s} \lesssim e^{K^{\frac{2}{\theta}} T} \|u_0\|_{H^s},$$

for any $T > 0$ and any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{L^2} \leq K$. \square

⁵ In the focusing case, one uses the Gagliardo-Nirenberg inequality (1.4), as discussed in the introduction.

References

- [1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I: Schrödinger equations, *Geom. Funct. Anal.* 3 (1993) 107–115.
- [2] J. Bourgain, Gibbs measures and quasi-periodic solutions for nonlinear Hamiltonian partial differential equations, in: *The Gelfand Mathematical Seminars, 1993–1995*, in: *Gelfand Math. Sem.*, Birkhäuser Boston, Boston, MA, 1996, pp. 23–43.
- [3] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, *Commun. Math. Phys.* 166 (1) (1994) 1–26.
- [4] J. Bourgain, Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity, *Int. Math. Res. Not.* (5) (1998) 253–283.
- [5] N. Burq, P. Gérard, N. Tzvetkov, An instability property of the nonlinear Schrödinger equation on S^d , *Math. Res. Lett.* 9 (2002) 323–335.
- [6] D. Cai, A. Majda, D. McLaughlin, E. Tabak, Dispersive wave turbulence in one dimension, *Physica D* 152/153 (2001) 551–572.
- [7] R. Cameron, W. Martin, Transformations of Wiener integrals under translations, *Ann. Math.* (2) 45 (1944) 386–396.
- [8] Y. Cho, G. Hwang, S. Kwon, S. Lee, Well-posedness and ill-posedness for the cubic fractional Schrödinger equations, *Discrete Contin. Dyn. Syst.* 35 (2015) 2863–2880.
- [9] A. Choffrut, O. Pocovnicu, Ill-posedness of the cubic nonlinear half-wave equation and other fractional NLS on the real line, *Int. Math. Res. Not.* (3) (2018) 699–738.
- [10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, A refined global well-posedness result for Schrödinger equations with derivative, *SIAM J. Math. Anal.* 34 (1) (2002) 64–86.
- [11] S. Demirbas, M.B. Erdoğan, N. Tzirakis, Existence and Uniqueness theory for the fractional Schrödinger equation on the torus, in: *Some Topics in Harmonic Analysis and Applications*, in: *Advanced Lectures in Mathematics (ALM)*, vol. 34, Int. Press, Somerville, MA, 2016, pp. 1241–1252.
- [12] A. Elgart, B. Schlein, Mean field dynamics of boson stars, *Commun. Pure Appl. Math.* 60 (4) (2007) 500–545.
- [13] M.B. Erdoğan, T.B. Gürel, N. Tzirakis, Smoothing for the fractional Schrödinger equation on the torus and the real line, *Indiana Univ. Math. J.* 68 (2019) 369–392.
- [14] J. Fröhlich, E. Lenzmann, Blowup for nonlinear wave equations describing boson stars, *Commun. Pure Appl. Math.* 60 (11) (2007) 1691–1705.
- [15] P. Gérard, An introduction to the cubic Szegő equation, in: *Lectures on the Analysis of Nonlinear Partial Differential Equations. Part 3*, in: *Morningside Lect. Math.*, vol. 3, Int. Press, Somerville, MA, 2013, pp. 177–208.
- [16] P. Gérard, E. Lenzmann, O. Pocovnicu, P. Raphaël, A two-soliton with transient turbulent regime for the cubic half-wave equation on the real line, *Ann. PDE* 4 (1) (2018), Art. 7, 166 pp.
- [17] P. Gérard, S. Grellier, Effective integrable dynamics for a certain nonlinear wave equation, *Anal. PDE* 5 (5) (2012) 1139–1155.
- [18] J. Ginibre, Y. Tsutsumi, G. Velo, On the Cauchy problem for the Zakharov system, *J. Funct. Anal.* 151 (1997) 384–436.
- [19] T.S. Gunaratnam, T. Oh, N. Tzvetkov, H. Weber, Quasi-invariant Gaussian measures for the nonlinear wave equation in three dimensions, *arXiv:1808.03158v2 [math.PR]*.
- [20] Z. Guo, T. Oh, Non-existence of solutions for the periodic cubic nonlinear Schrödinger equation below L^2 , *Int. Math. Res. Not.* (6) (2018) 1656–1729.
- [21] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, fifth edition, The Clarendon Press, Oxford University Press, New York, 1979, xvi+426 pp.
- [22] B.A. Ivanov, A.M. Kosevich, Stable three-dimensional small-amplitude soliton in magnetic materials, *Sov. J. Low Temp. Phys.* 9 (1983) 439–442.
- [23] K. Kirkpatrick, E. Lenzmann, G. Staffilani, On the continuum limit for discrete NLS with long-range interactions, *Commun. Math. Phys.* 317 (2013) 563–591.
- [24] N. Kishimoto, A remark on norm inflation for nonlinear Schrödinger equations, *Commun. Pure Appl. Anal.* 18 (2019) 1375–1402.
- [25] J. Krieger, E. Lenzmann, P. Raphaël, Nondispersive solutions to the L^2 -critical half-wave equation, *Arch. Ration. Mech. Anal.* 209 (1) (2013) 61–129.
- [26] A. Majda, A. McLaughlin, E. Tabak, A one-dimensional model for dispersive wave turbulence, *J. Nonlinear Sci.* 7 (1) (1997) 9–44.
- [27] L. Molinet, On ill-posedness for the one-dimensional periodic cubic Schrödinger equation, *Math. Res. Lett.* 16 (1) (2009) 111–120.
- [28] A. Nahmod, L. Rey-Bellet, S. Sheffield, G. Staffilani, Absolute continuity of Brownian bridges under certain gauge transformations, *Math. Res. Lett.* 18 (5) (2011) 875–887.
- [29] T. Ogawa, Y. Tsutsumi, Blow-Up Solutions for the Nonlinear Schrödinger Equation with Quartic Potential and Periodic Boundary Conditions, *Springer Lecture Notes in Math.*, vol. 1450, 1990, pp. 236–251.
- [30] T. Oh, A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces, *Funkc. Ekvacioj* 60 (2017) 259–277.
- [31] T. Oh, P. Sosoe, N. Tzvetkov, An optimal regularity results on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation, *J. Éc. Polytech. Math.* 5 (2018) 793–841.
- [32] T. Oh, Y. Tsutsumi, N. Tzvetkov, Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third order dispersion, *C. R. Math. Acad. Sci. Paris* 357 (4) (2019) 366–381.
- [33] T. Oh, N. Tzvetkov, Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation, *Probab. Theory Relat. Fields* 169 (3–4) (2017) 1121–1168.
- [34] T. Oh, N. Tzvetkov, On the transport of Gaussian measures under the flow of Hamiltonian PDEs, in: *Séminaire Laurent Schwartz-Équations aux dérivées partielles et applications. Année 2015–2016, Exp. No. VI*, Ed. Éc. Polytech., Palaiseau, 2017, 9 pp.
- [35] T. Oh, N. Tzvetkov, Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation, *J. Eur. Math. Soc.* (2019), in press.

- [36] T. Oh, N. Tzvetkov, Y. Wang, Solving the 4NLS with white noise initial data, arXiv:1902.06169 [math.AP].
- [37] T. Oh, Y. Wang, On the ill-posedness of the cubic nonlinear Schrödinger equation on the circle, *An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat.* 64 (1) (2018) 53–84.
- [38] F. Planchon, N. Tzvetkov, N. Visciglia, Transport of Gaussian measures by the flow of the nonlinear Schrödinger equation, arXiv:1810.00526 [math.AP].
- [39] R. Ramer, On nonlinear transformations of Gaussian measures, *J. Funct. Anal.* 15 (1974) 166–187.
- [40] C. Sulem, P.L. Sulem, *The Nonlinear Schrödinger Equations: Self-focusing and Wave Collapse*, Applied Mathematical Sciences, vol. 139, Springer-Verlag, New York, 1999, 350 pp.
- [41] J. Thirouin, On the growth of Sobolev norms of solutions of the fractional defocusing NLS equation on the circle, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 34 (2) (2017) 509–531.
- [42] S.K. Turitsyn, Three-dimensional dispersion of nonlinearity and stability of multidimensional solitons, *Teor. Mat. Fiz.* 64 (1985) 226–232 (in Russian).
- [43] N. Tzvetkov, Quasi-invariant Gaussian measures for one-dimension Hamiltonian partial differential equations, *Forum Math. Sigma* 3 (e28) (2015) 35.
- [44] P. Zhidkov, On an infinite sequence of invariant measures for the cubic nonlinear Schrödinger equation, *Int. J. Math. Math. Sci.* 28 (7) (2001) 375–394.